LECTURE

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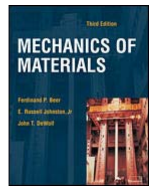
BEAMS: STATICALLY INDETERMINATE

18

Chapter 9.5

A. J. Clark School of Engineering •Department of Civil and Environmental Engineering



by

Dr. Ibrahim A. Assakkaf

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ENES 220 – Mechanics of Materials

Department of Civil and Environmental Engineering

University of Maryland, College Park

Statically Indeterminate Beams

- Introduction
  - Our previous analysis was limited to statically determinate beams.
  - A beam, subjected only to transverse loads, with more than two reaction components, is statically indeterminate because the equations of equilibrium are not sufficient to determine all the reactions.

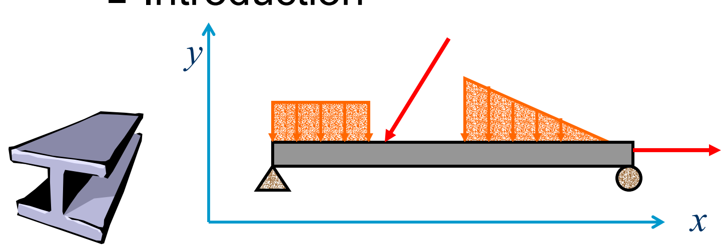
Statically Indeterminate Beams

- Introduction
  – In all of the problems discussed so far, it was possible to determine the forces and stresses in beams by utilizing the equations of equilibrium, that is

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_A = 0 \tag{29}$$

Statically Indeterminate Beams

- Introduction



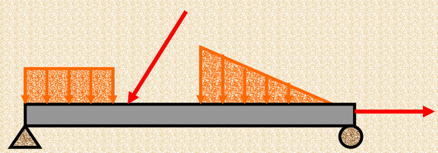
Coplanar Force System

$$\sum F_x = 0 \qquad \sum F_y = 0$$
$$\sum M_A = 0$$

Statically Indeterminate Beams

- Statically Determinate Beam

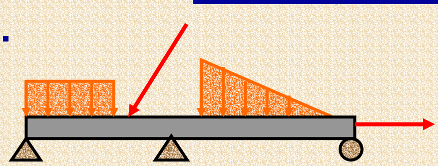
  When the equations of equilibrium are sufficient to determine the forces and stresses in a structural beam, we say that this beam is *statically determinate*



Statically Indeterminate Beams

- Statically Indeterminate Beam

  When the equilibrium equations alone are not sufficient to determine the loads or stresses in a beam, then such beam is referred to as *statically indeterminate* beam.



Statically Indeterminate Beams

- Determinacy of Beams
  – For a coplanar (two-dimensional) beam, there are at most three equilibrium equations for each part, so that if there is a total of *n* parts and *r* reactions, we have

$$\boxed{\begin{aligned} r = 3n, &\quad \Rightarrow \text{statically determinate} \\ r > 3n, &\quad \Rightarrow \text{statically indeterminate} \end{aligned}} \quad (30)$$

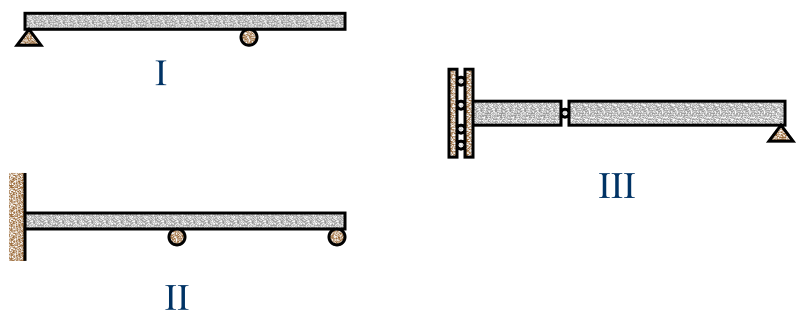
Statically Indeterminate Beams

- Example 11

  Classify each of the beams shown as statically determinate of statically indeterminate. If statically indeterminate, report the degrees of of determinacy. The beams are subjected to external loadings that are assumed to be known and can act anywhere on the beams.

Statically Indeterminate Beams

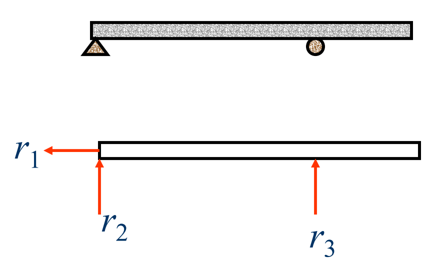
- Example 11 (cont'd)



Statically Indeterminate Beams

- Example 11 (cont'd)

For part I:



Applying Eq. 30,

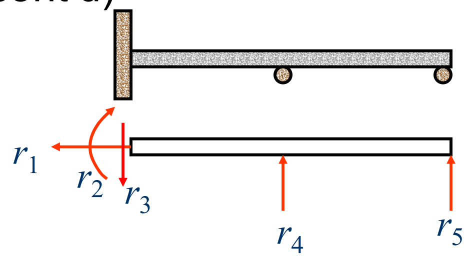
$r = 3,\ n = 1$, therefore,

$r = 3n, \Rightarrow 3 = [3(1) = 3] \Rightarrow$ statically determinate

Statically Indeterminate Beams

- Example 11 (cont'd)

For part II:



Applying Eq. 30,

$r = 5, n = 1$, therefore,

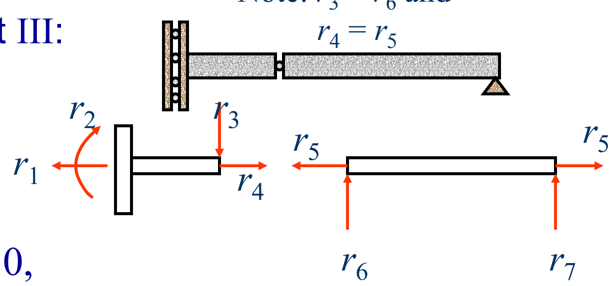
$r > 3n, \Rightarrow 5 > [3(1) > 3] \Rightarrow$ statically indeterminate to second degree

Statically Indeterminate Beams

- Example 11 (cont'd)

For part III:

Note: $r_3 = r_6$ and $r_4 = r_5$



Applying Eq. 30,

$r = 6, n = 2$, therefore,

$r = 3n, \Rightarrow 6 = [3(2) = 6] \Rightarrow$ statically determinate

Statically Indeterminate Transversely Loaded Beams

- *How to determine forces and stresses of transversely loaded beam that is statically indeterminate?*
  - In order to solve for the forces, and stresses in such beam, it becomes necessary to supplement the equilibrium equations with additional relationships based on any conditions of restraint that may exist.

Statically Indeterminate Transversely Loaded Beams

- In such cases the geometry of the deformation of the loaded beam is used to obtain the additional relations needed for an evaluation of the reactions (or other unknown forces).
- For problems involving elastic action, each additional constraint on a beam provides additional information concerning slopes or deflections.

Statically Indeterminate Transversely Loaded Beams

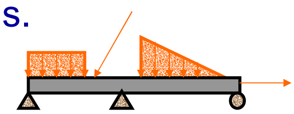
- Such information, when used with appropriate slope or deflection equations, yields expressions that supplement the independent equations of equilibrium.
- Finding the deflection curve for statically indeterminate beams requires no new theories or techniques.
- The unknown external reactions may be treated simply as ordinary external loads.

Statically Indeterminate Transversely Loaded Beams

- The deflection caused by these external loads can be found by any of the methods previously discussed: *direct integration*, *singularity functions*, *superposition*, or by use of beam *deflection tables*.
- Since the presence of the external reactions places geometrical restrictions on the deflection curve, there will always be a sufficient number of boundary conditions to find the unknown reactions.

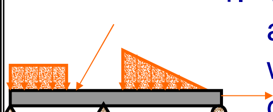
Statically Indeterminate Transversely Loaded Beams

- General Rules
  - Each statically indeterminate beam problem has its own peculiarities as to its method of solution.
  - But there are some general rules and ideas that are common to the solution of most types of beam problems.



Statically Indeterminate Transversely Loaded Beams

- General Rules (cont'd)
  - These general rules and guidelines are summarized as follows:
    1. Write the appropriate equations of equilibrium and examine them carefully to make sure whether or not the beam problem is statically determinate or indeterminate. Eq. 30 can help in the case of coplanar problems.
    2. If the problem is statically indeterminate, examine the kinematic restraints to determine

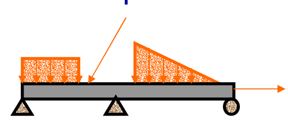


Statically Indeterminate Transversely Loaded Beams

- General Rules (cont'd)

  the necessary conditions that must be satisfied by the deformation of the beam.

  3. Express the required deformations in terms of the loads or forces. When enough of these additional relationships have been obtained, they can be adjoined to the equilibrium equations and the beam problem can then be solved.



Statically Indeterminate Transversely Loaded Beams

- The Integration Method
  – Consider the simply supported cantilever beam that is subjected to a uniformly distributed load *w*

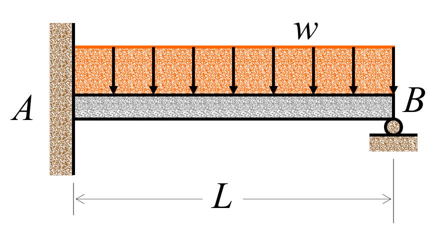


Figure 35

Statically Indeterminate Transversely Loaded Beams

- The Integration Method
  - Drawing the free-body diagram of the beam (see Fig. 36), we note that the reactions involve four unknowns, while only three equilibrium equations are available, namely

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_A = 0 \qquad (31)$$

Statically Indeterminate Transversely Loaded Beams

- The Integration Method

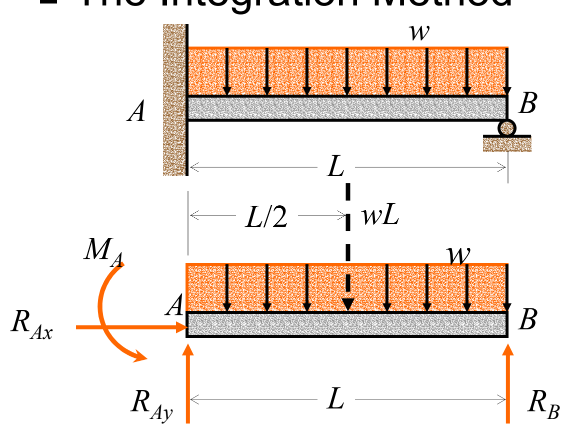
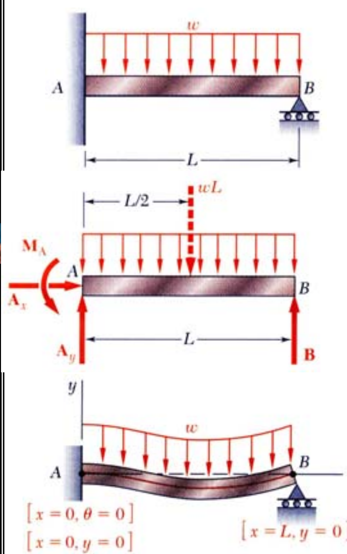


Figure 36

Statically Indeterminate Beams



- Consider beam with fixed support at A and roller support at B.
- From free-body diagram, note that there are four unknown reaction components.
- Conditions for static equilibrium yield

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

The beam is statically indeterminate.

- Also have the beam deflection equation,

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

which introduces two unknowns but provides three additional equations from the boundary conditions:

At $x = 0,\ \theta = 0\ y = 0$ At $x = L,\ y = 0$

Statically Indeterminate Transversely Loaded Beams

- The Integration Method
  - The problem is obviously indeterminate to the first degree because we have three unknown reactions and only three equations of equilibrium.
  - We know that in statically indeterminate problem, the reactions may be obtained by considering the deformation of the structure involved.

Statically Indeterminate Transversely Loaded Beams

- The Integration Method
  – We should, therefore, proceed with the computation of the slope and deformation along the beam.
  – First, the bending moment $M(x)$ at any given point of beam AB is expressed in terms of the distance x from A, the given load, and the unknown reactions.

Statically Indeterminate Transversely Loaded Beams

- The Integration Method
  – *Integrating* in x, expressions for the slope θ and the deflection y, which contain two additional unknowns, namely, the constants of integration C_1 and C_2.
  – But altogether six equations are available to determine the reactions and the constants C_1 and C_2.

Statically Indeterminate Transversely Loaded Beams

- The Integration Method
  - These six equations are:
    - The three equilibrium equations (Eq. 31)
    - The three equations expressing that the boundary conditions are satisfied, i.,e., that slope and deflection at *A* are zero, and that the deflection at *B* is zero (Fig. 37).
  - Thus the reactions at the supports may be determined, and the equations for the elastic curve may be obtained.

Statically Indeterminate Transversely Loaded Beams

- The Integration Method

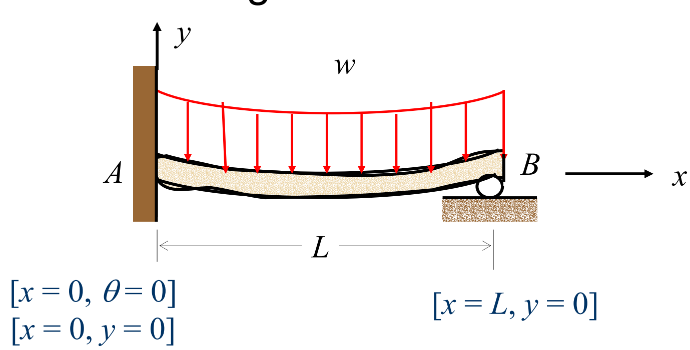
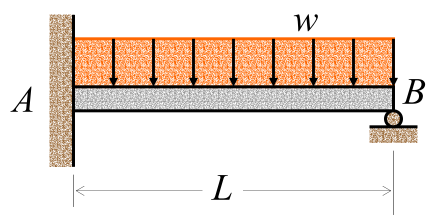


Figure 37. Deflected Shape of the Beam and the Boundary Conditions

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method
  - Determine the reactions at the supports for the simply supported cantilever beam of Figure 35 in terms of *w* and *L*.



Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method
  - **Equilibrium Equations:**
    - From the free body diagram of Fig. 38, we write

$$+\rightarrow \sum F_x = 0; \quad R_{A_x} = 0 \qquad (32a)$$

$$+\uparrow \sum F_y = 0; \quad R_{A_y} + R_B - wL = 0 \qquad (32b)$$

$$+\curvearrowright \sum M_A = 0; \quad -M_A - R_B L + \frac{1}{2} wL^2 = 0 \qquad (32c)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)

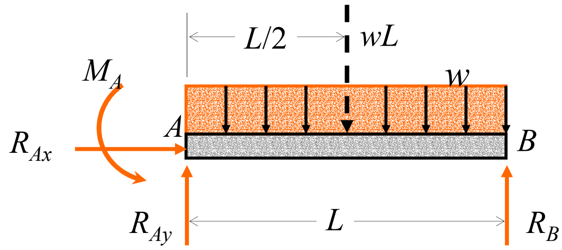


Figure 38. Free-body Diagram for the Beam

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)
  - **Equation of Elastic Curve:**
    - Drawing the free-body diagram of a portion of the beam (*AC*) as shown in Fig. 39, we write

$$+\circlearrowleft \sum M_C = 0; \quad -M(x) - \frac{1}{2}wx^2 - M_A + R_{A_y}x = 0$$

or

$$M(x) = -\frac{1}{2}wx^2 + R_{A_y}x - M_A \qquad (33)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)

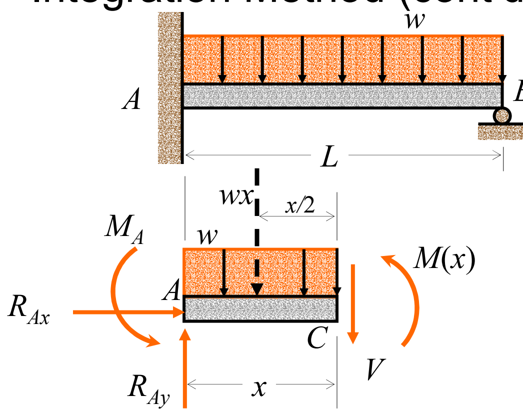


Figure 38.
Free-body Diagram for the portion *AC* of the Beam

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)
  - Equating the expression for $M(x)$ of Eq.33, to the curvature times EI , and integrating twice, gives

$$EIy'' = -\frac{1}{2}wx^2 + R_{A_y}x - M_A \quad \text{(34a)}$$

$$EI\theta = EIy' = -\frac{1}{6}wx^3 + \frac{1}{2}R_{A_y}x^2 - M_A x + C_1 \quad \text{(34b)}$$

$$EIy = -\frac{1}{24}wx^4 + \frac{1}{6}R_{A_y}x^3 - \frac{1}{2}M_A x^2 + C_1 x + C_2 \quad \text{(34c)}$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)
  - Referring to boundary conditions shown in Fig. 37, we make $x = 0$, $\theta = 0$ in Eq. 34b, $x = 0$, $y = 0$ in Eq. 34c, and conclude that $C_1 = C_2 = 0$.
  - Thus, Eq. 34c can be rewritten as follows to the elastic curve expression:

$$EIy = -\frac{wx^4}{24} + \frac{R_{A_y} x^3}{6} - \frac{M_A x^2}{2} \qquad (35)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)
  - But the third boundary condition requires that $y = 0$ for $x = L$. Therefore, substituting these values into Eq. 35, gives

$$EIy(0) = 0 = -\frac{wL^4}{24} + \frac{R_{A_y} L^3}{6} - \frac{M_A L^2}{2}$$

or

$$3M_A - R_{A_y} + \frac{wL^2}{4} = 0 \qquad (36)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using the Integration Method (cont'd)
  - Solving this equation simultaneously with the three equilibrium equations (Eq. 32), the reactions at the supports are determined as follows:

$$\boxed{R_{A_x} = 0}$$

$$\boxed{R_{A_y} = \frac{5}{8}wL}$$

$$\boxed{M_A = \frac{1}{8}wL^2}$$

$$\boxed{R_B = \frac{3}{8}wL}$$