LECTURE

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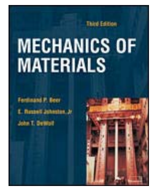
Third Edition

BEAMS: BENDING STRESS

9

Chapter 4.1 – 4.5 4.13

A. J. Clark School of Engineering •Department of Civil and Environmental Engineering



by
Dr. Ibrahim A. Assakkaf

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ENES 220 – Mechanics of Materials

Department of Civil and Environmental Engineering
University of Maryland, College Park

Beams

- Introduction
  - The most common type of structural member is a beam.
  - In actual structures beams can be found in an infinite variety of
    - Sizes
    - Shapes, and
    - Orientations

Beams

- Introduction

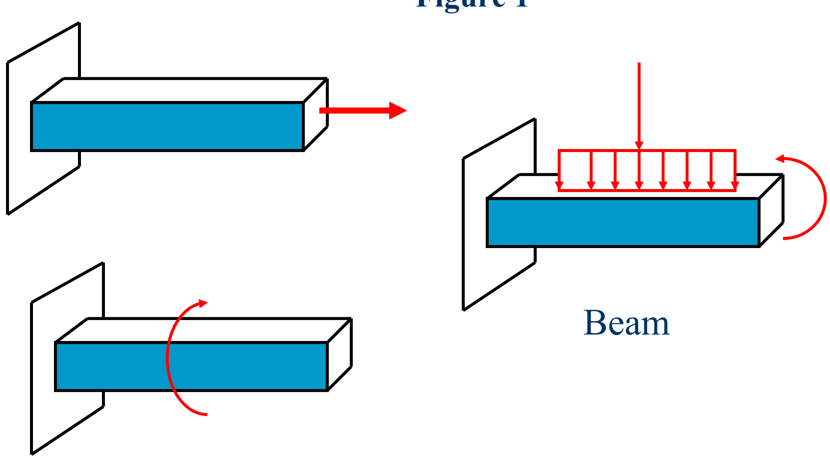
Definition

A beam may be defined as a member whose length is relatively large in comparison with its thickness and depth, and which is loaded with transverse loads that produce significant bending effects as oppose to twisting or axial effects

Beams

- Introduction

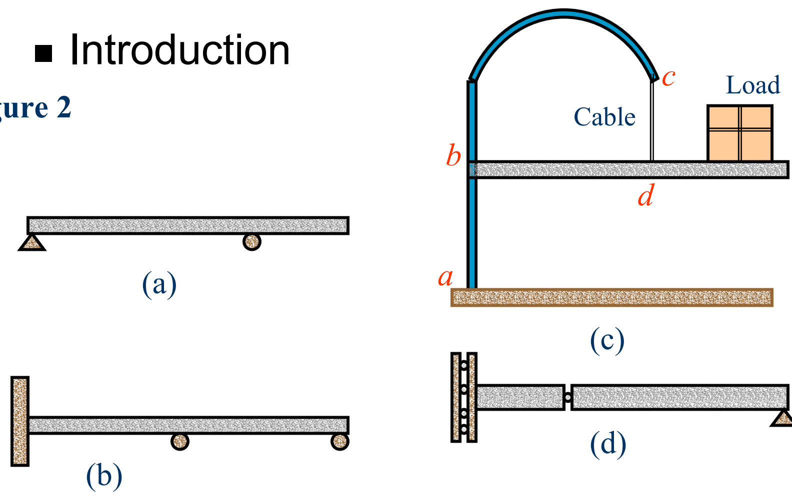
Figure 1



Beams

■ Introduction

Figure 2



Beams

■ Introduction
- Beams can be
  - Straight as shown in Figure 1c
    - For example the straight member *bde*
  - Curved as shown in Figure 1c
    - For example the curved member *abc*

Beams

- Introduction
  - Beams are generally classified according to their geometry and the manner in which they are supported.
  - Geometrical classification includes such features as the shape of the cross section, whether the beam is
    - *straight* or
    - *curved*

Beams

- Introduction
  - Or whether the beam is
    - *Tapered*, or
    - Has a *constant cross section*.
  - And some other features that will be discussed later

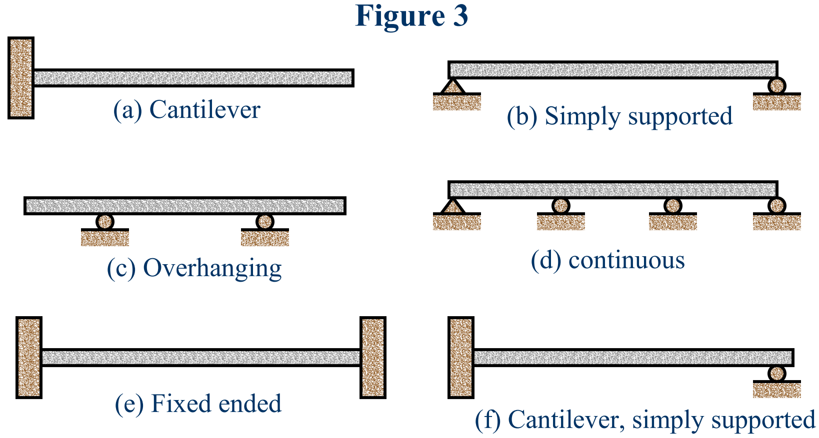
Beams

- Introduction
  - Beams can also be classified according to the manner in which they are supported. Some types that occur in ordinary practice are shown in Figure 3, the names of some of these being fairly obvious from direct observation.
  - Note that the beams in (d), (e), and (f) are statically indeterminate.

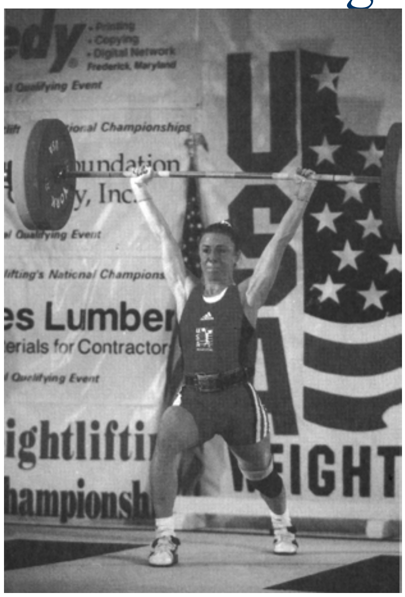
Beams

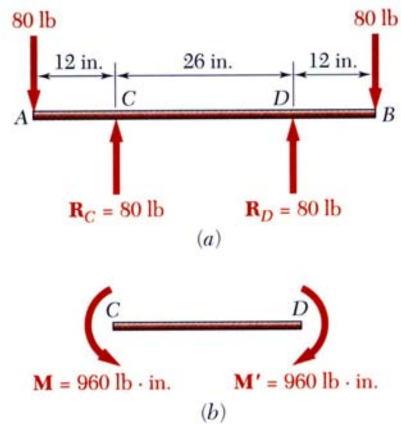
- Introduction

Figure 3



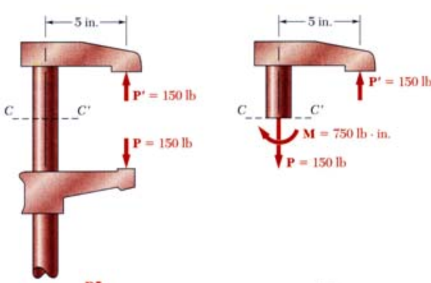
Pure Bending

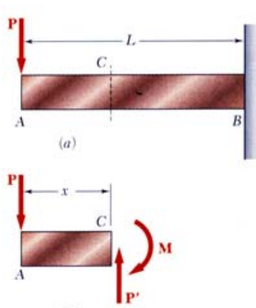




Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

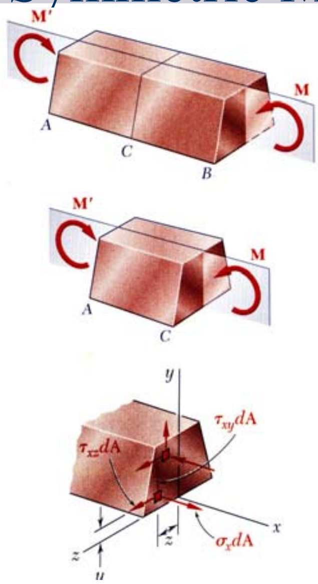
Other Loading Types





- *Eccentric Loading*: Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple
- *Transverse Loading*: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple
- *Principle of Superposition*: The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.

Symmetric Member in Pure Bending



- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x \, dA = 0$$
$$M_y = \int z\sigma_x \, dA = 0$$
$$M_z = \int -y\sigma_x \, dA = M$$

Normal and Shearing Stress

- Normal Stress σ in beams

  The normal stress on plane *a-a* is related to the resisting moment M_r as follows (see Figure 4):

$$M_r = -\int_{\text{area}} y\sigma \, dA \qquad (1)$$

Normal and Shearing Stress

- Stresses in beams

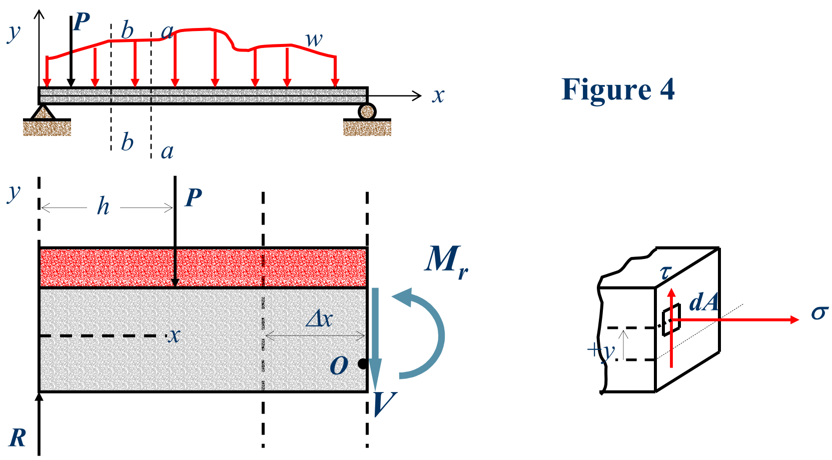


Figure 4

Normal and Shearing Stress

- Shearing Stress τ in beams

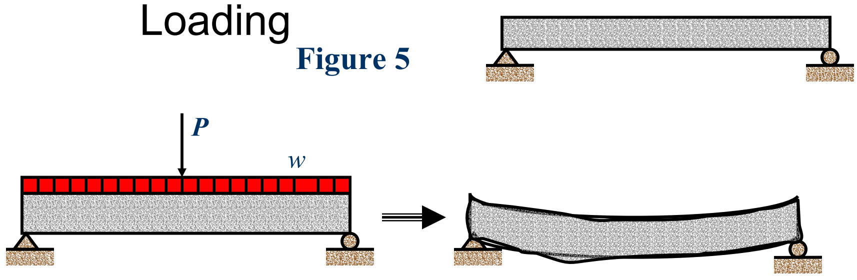
  The shearing stress on plane *a-a* is related to the resisting shear *V* as follows (see Figure 4):

$$V_r = -\int_{\text{area}} \tau \, dA \qquad (2)$$

Flexural Strains

- Deformation of Beam due to Lateral Loading

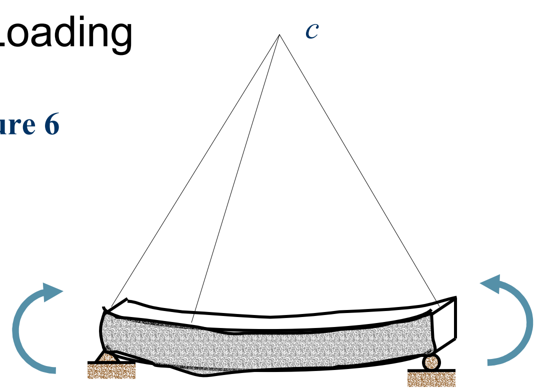
Figure 5



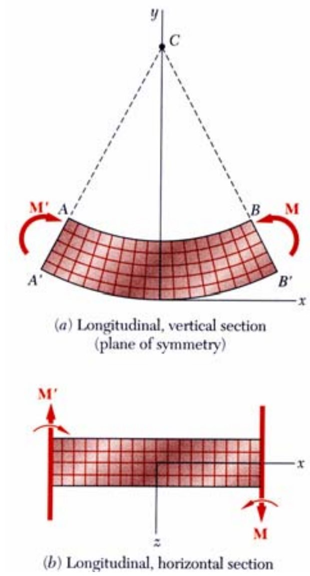
Flexural Strains

- Deformation of Beam due to Lateral Loading

Figure 6



Bending Deformations



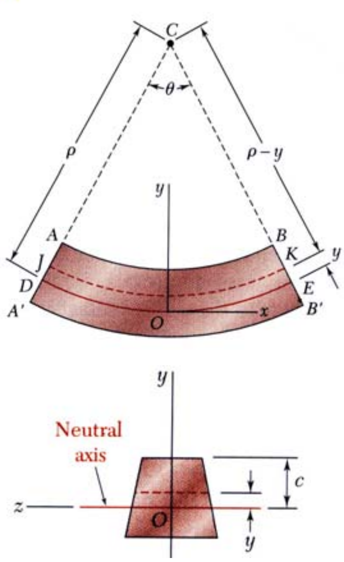
(a) Longitudinal, vertical section (plane of symmetry)

(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending



Consider a beam segment of length L.

After deformation, the length of the neutral surface remains L. At other sections,

$$L' = (\rho - y)\theta$$

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Flexural Strains

- Deformation of Beam due to Lateral Loading
  – A segment of the beam of Fig. 4 between planes *a-a* and *b-b* is shown in Figure 7 with the deformation (distortion) is greatly exaggerated
  – Assumption is made that a plane section before bending remains plane after bending.

Flexural Strains

- Deformation of Beam due to Lateral Loading

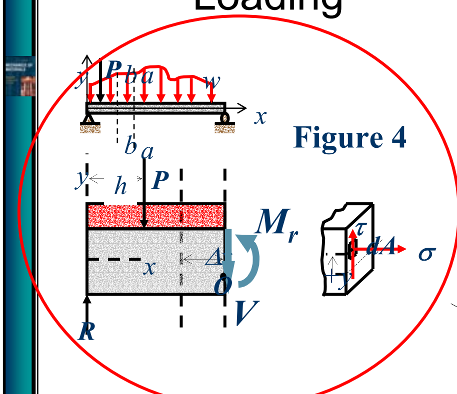


Figure 4

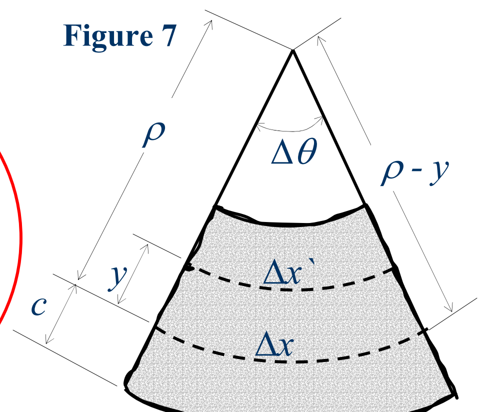


Figure 7

Flexural Strains

- Stresses in beams

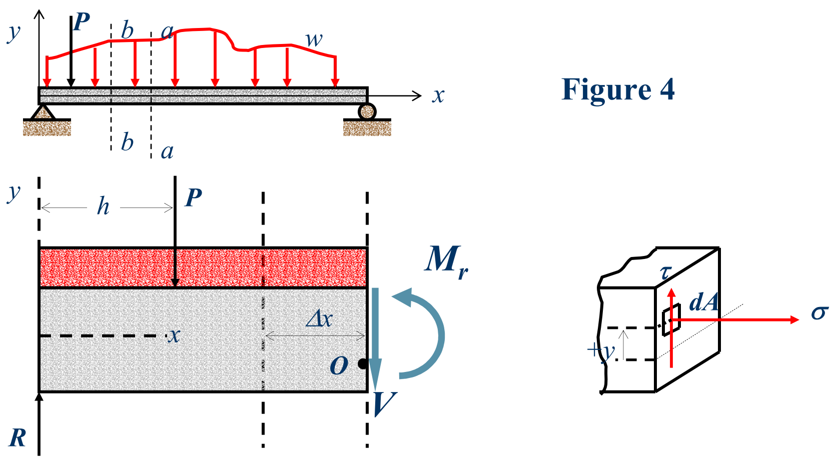


Figure 4

Flexural Strains

- Experimental Results
  - Precise experimental measures suggest that at some distance *c* (see Figure 7) above the bottom of the beam, longitudinal elements undergo no change in length.
  - The curved surface formed by these elements (at radius ρ) is referred to as the neutral surface of the beam, and the intersection of this surface with any cross section is called the *neutral axis of the section*.

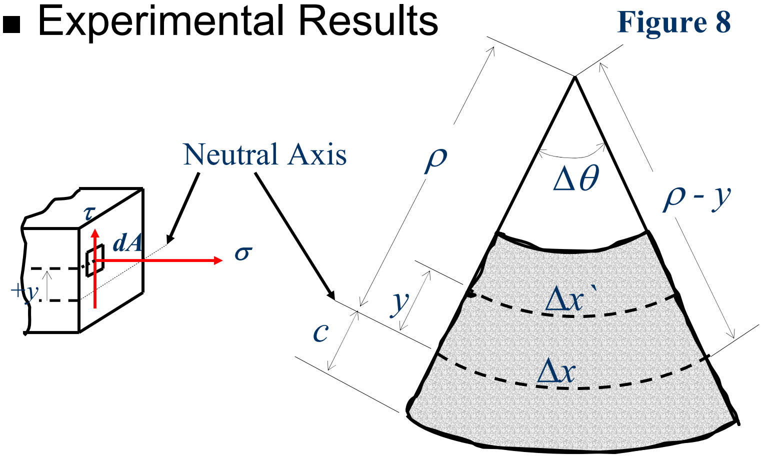
Flexural Strains

- Experimental Results
  - All elements (fibers) on one side of the neutral surface are compressed and those on the opposite side are elongated.
  - In reference to Fig. 8, the fibers above the neutral surface of the beam are compressed, while those below the neutral axis are elongated.

Flexural Strains

- Experimental Results

Figure 8



Flexural Strains

- Longitudinal Strain
  – The longitudinal strain ε_x experienced by a longitudinal element that is located a distance *y* from the neutral axis (surface) of the beam is determined as follows:

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{L_f - L_i}{L_f} \qquad (3)$$

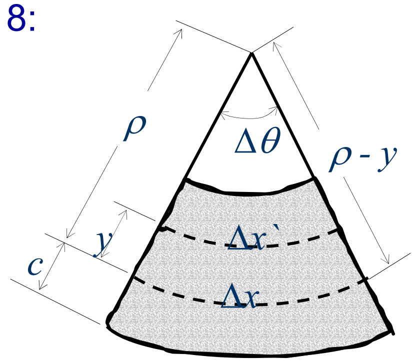
L_f = final length after loading
L_i = initial length before loading

Flexural Strains

- Longitudinal Strain
  – From the geometry of the beam as shown in Figure 7 and 8:

$$\varepsilon_x = \frac{\Delta x' - \Delta x}{\Delta x} \qquad (4)$$

$$= \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$



Flexural Strains

- Longitudinal Strain

$$\varepsilon_x = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

$$= \frac{\rho\Delta\theta - y\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} = \frac{-y\Delta\theta}{\rho\Delta\theta}$$

$$\varepsilon_x = -\frac{1}{\rho}y \qquad (5)$$

Flexural Strains

- Longitudinal Strain

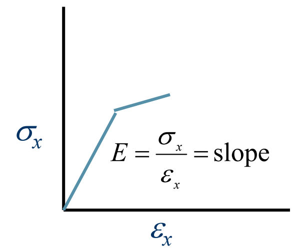
  The normal longitudinal strain ε_x varies linearly, through the member, with the distance y from the neutral surface, and it is given by

$$\varepsilon_x = -\frac{y}{\rho} \qquad (6)$$

Flexural Stress

- For special case of linearly elastic deformation, the relationship between the normal stress σ_x and the normal strain ε_x is given by Hooke's law as

$$E = \frac{\sigma_x}{\varepsilon_x} = \text{slope} \qquad (7)$$



Flexural Stress

- Flexural Normal Stress
  - Eq. 7 can be rewritten as

$$\sigma_x = \varepsilon_x E \qquad (8)$$

  - Recall Eq. (6) $\varepsilon_x = -\frac{y}{\rho}$, therefore

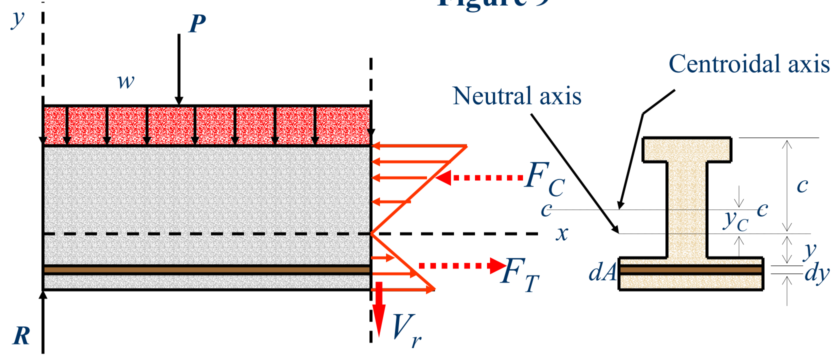
$$\sigma_x = \varepsilon_x E = -\frac{y}{\rho} E \qquad (9)$$

Flexural Stress

- Flexural Normal Stress

Distribution of Normal Stress in a Beam Cross Section

Figure 9



Flexural Stress

- Flexural Normal Stress
  - The resisting moment M_r that can be develop by the normal stress in a typical beam with loading in a plane of symmetry but arbitrary cross section (Fig. 9) is given by Eq. 1 as

$$M_r = -\int_{\text{area}} y\sigma_x \, dA$$

Flexural Stress

- Flexural Normal Stress
  - Since *y* is measured from the neutral axis (surface), it is necessary to locate this axis by means of the equilibrium equation as follows:

$$\sum F_x = 0$$

$$\int_A \sigma_x dA = 0 \qquad (10)$$

Flexural Stress

- Flexural Normal Stress
  - Substituting for σ_x given by Eq. 9 into Eq. 10, yields

$$\begin{aligned}\int_A \sigma_x dA &= \int_A -\frac{y}{\rho} E dA \\ &= -\frac{E}{\rho}\int_A y dA = 0\end{aligned} \qquad (11)$$

  - But

$$y_C = \int_A y dA = \text{distance from neutral axis to centroidal axis } (c\text{-}c) \qquad (12)$$

Flexural Stress

- Flexural Normal Stress
  - Therefore, Eq. 11 becomes

$$\int_A \sigma_x dA = -\frac{E}{\rho} y_C A = 0 \qquad (12)$$

  - Since neither (E/ρ) nor A are zero, y_C must equal zero.

Flexural Stress

- Flexural Normal Stress

For flexural loading and linearly elastic action, the neutral axis passes through the centroid of the cross section of the beam

Flexural Stress

- Flexural Normal Stress
  - The maximum normal stress on the cross section is given by

$$\sigma_{\max} = -\frac{E}{\rho}c \qquad (13)$$

  - Combining Eqs. 9 and 13, hence

$$\sigma_x = \frac{y}{c}\sigma_{\max} = \frac{y}{c}\sigma_c \qquad (14)$$

Flexural Stress

- Flexural Normal Stress
  - Substituting Eq. 14 into Eq. 1, gives

$$M_r = -\int_{\text{area}} y\sigma_x \, dA$$

$$= -\int_A y\left(\frac{y}{c}\sigma_c\right)dA = -\frac{\sigma_c}{c}\int_A y^2 dA \qquad (15)$$

  - The integral $\int y^2 dA$ is called the second moment of area, and it is given the symbol I.

Flexural Stress

- Flexural Normal Stress
  – Substituting for the second moment of area I of the cross section of the beam into Eq. 15, yields

$$M_r = -\frac{\sigma_c}{c}\int_A y^2 dA = -\frac{\sigma_c}{c} I \qquad (16)$$

or

$$\sigma_c = \frac{M_r}{I} c = \sigma_{\max} \qquad (17)$$

Elastic Flexural Formula

- The elastic flexural formula for normal stress is given by

$$\sigma_{\max} = \frac{M_r c}{I} \qquad (18)$$

and

$$\sigma_x = \frac{M_r y}{I} \qquad (19)$$

Elastic Flexural Formula

- An alternative form of the flexural formula for maximum normal stress is given by

$$\sigma_{\max} = \frac{M_r}{S} \qquad (20)$$

Where

$$S = \frac{I}{c}$$

Stress Due to Bending

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

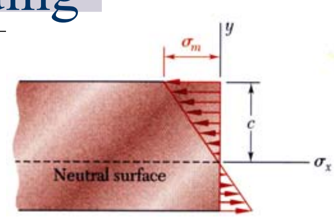
$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



- For static equilibrium,

$$M = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) dA$$

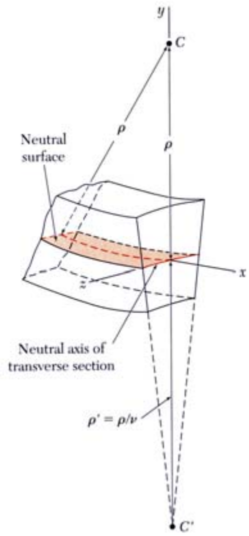
$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$

Deformation in a Symmetric Member in Pure Bending



- Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$
$$= \frac{M}{EI}$$

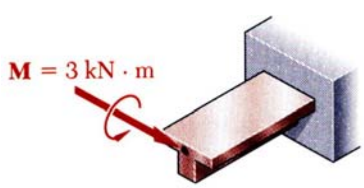
- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

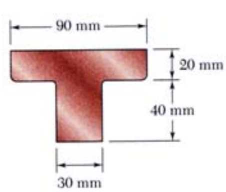
$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvature}$$

Example 1





A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing E = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + A d^2\right)$$

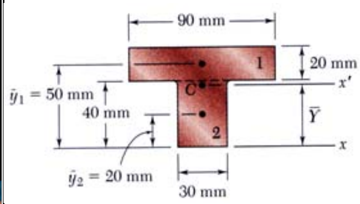
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

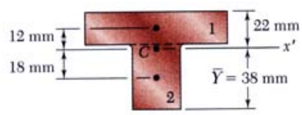
Example 1 (cont'd)



SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

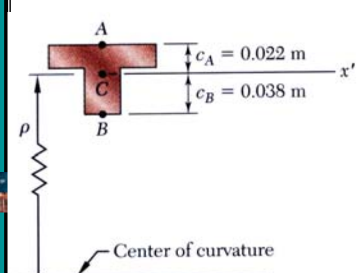
	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20\times 90 = 1800$	50	90×10^3
2	$40\times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114\times 10^3$



$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114\times 10^3}{3000} = 38\,\text{mm}$$

$$I_{x'} = \sum\left(\bar{I} + A d^2\right) = \sum\left(\tfrac{1}{12} bh^3 + A d^2\right)$$
$$= \left(\tfrac{1}{12} 90\times 20^3 + 1800\times 12^2\right) + \left(\tfrac{1}{12} 30\times 40^3 + 1200\times 18^2\right)$$
$$I = 868\times 10^3\,\text{mm} = 868\times 10^{-9}\,\text{m}^4$$

Example 1 (cont'd)



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

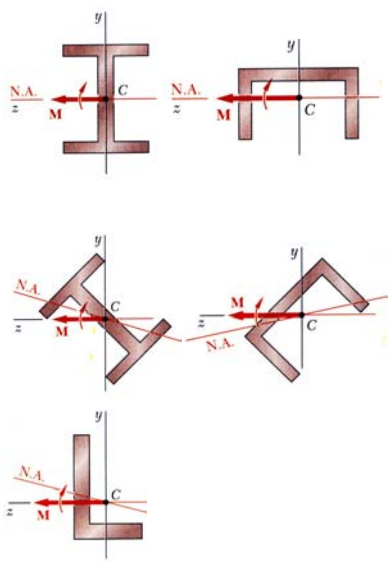
$$\sigma_m = \frac{Mc}{I}$$
$$\sigma_A = \frac{M c_A}{I} = \frac{3\,\text{kN}\cdot\text{m}\times 0.022\,\text{m}}{868\times 10^{-9}\,\text{mm}^4} \qquad \sigma_A = +76.0\,\text{MPa}$$
$$\sigma_B = -\frac{M c_B}{I} = -\frac{3\,\text{kN}\cdot\text{m}\times 0.038\,\text{m}}{868\times 10^{-9}\,\text{mm}^4} \qquad \sigma_B = -131.3\,\text{MPa}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$
$$= \frac{3\,\text{kN}\cdot\text{m}}{(165\,\text{GPa})\left(868\times 10^{-9}\,\text{m}^4\right)}$$

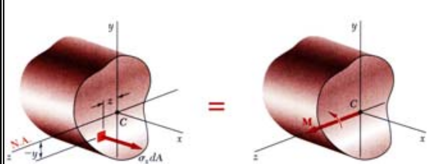
$$\frac{1}{\rho} = 20.95\times 10^{-3}\,\text{m}^{-1}$$
$$\rho = 47.7\,\text{m}$$

Unsymmetrical Bending



- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.

Unsymmetrical Bending



Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

  $F_x = 0 = M_y \quad M_z = M = applied\,couple$

- $0 = F_x = \int \sigma_x dA = \int \left(-\frac{y}{c}\sigma_m \right) dA$

  or $0 = \int y\, dA$

  neutral axis passes through centroid

- $M = M_z = -\int y \left(-\frac{y}{c}\sigma_m \right) dA$

  or $M = \frac{\sigma_m I}{c} \quad I = I_z = moment\ of\ inertia$

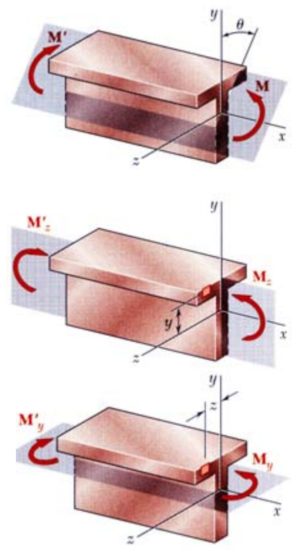
  defines stress distribution

- $0 = M_y = \int z \sigma_x dA = \int z \left(-\frac{y}{c}\sigma_m \right) dA$

  or $0 = \int yz\, dA = I_{yz} = product\ of\ inertia$

  couple vector must be directed along a principal centroidal axis

Unsymmetrical Bending



Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

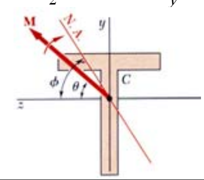
- Superpose the component stress distributions

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Along the neutral axis,

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} = -\frac{(M\cos\theta)y}{I_z} + \frac{(M\sin\theta)y}{I_y}$$

$$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$



Unsymmetrical Bending

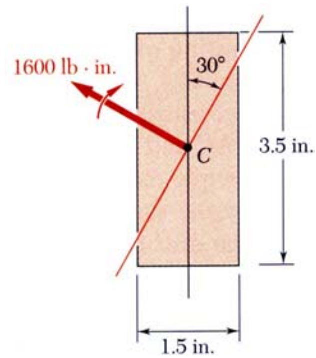
- The form of the flexural formula for normal stress in unsymmetrical bending is given by

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} \qquad (21)$$

Where

M_z, I_z = moment and moment of inertia about *z*-axis
M_y, I_y = moment and moment of inertia about *z*-axis

Example 2



A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

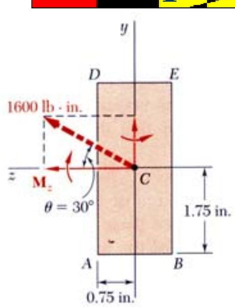
- Combine the stresses from the component stress distributions.

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Determine the angle of the neutral axis.

$$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$

Example 2 (cont'd)



$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (1600\,\text{lb}\cdot\text{in})\cos 30 = 1386\,\text{lb}\cdot\text{in}$$
$$M_y = (1600\,\text{lb}\cdot\text{in})\sin 30 = 800\,\text{lb}\cdot\text{in}$$
$$I_z = \tfrac{1}{12}(1.5\,\text{in})(3.5\,\text{in})^3 = 5.359\,\text{in}^4$$
$$I_y = \tfrac{1}{12}(3.5\,\text{in})(1.5\,\text{in})^3 = 0.9844\,\text{in}^4$$

The largest tensile stress due to M_z occurs along AB

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386\,\text{lb}\cdot\text{in})(1.75\,\text{in})}{5.359\,\text{in}^4} = 452.6\,\text{psi}$$

The largest tensile stress due to M_z occurs along AD

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800\,\text{lb}\cdot\text{in})(0.75\,\text{in})}{0.9844\,\text{in}^4} = 609.5\,\text{psi}$$

- The largest tensile stress due to the combined loading occurs at A.

$$\sigma_{max} = \sigma_1 + \sigma_2 = 452.6 + 609.5 \qquad \sigma_{max} = 1062\,\text{psi}$$