Stresses in Oblique Planes

- Up to this point, the stresses in a shaft has been limited to shearing stresses.
- This due to the fact that the selection of the element under study was oriented in such a way that its faces were either perpendicular or parallel to the axis of the shaft (see Fig. 15)
From our discussion of the torsional loading on a shaft, we know this loading produces shearing stresses $\tau$ in the faces perpendicular to the axis of the shaft.

But due to equilibrium requirement, there are equal stresses on the faces formed by the two planes containing the axis of the shaft.
Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - It is necessary to make sure that whether the transverse plane is a plane of maximum shearing stress and whether there are other significant stresses induced by torsion.
  - Consider the following shaft (Fig. 16), which is subjected to a torque $T$.

![Stresses in Oblique Planes](Image)

Fig. 16

- Other Stresses Induced By Torsion
Other Stresses Induced By Torsion

- The stresses at point A in the shaft of Fig. 16a is analyzed.
- A differential element taken from the shaft at point A and the stresses acting on transverse and longitudinal planes are shown in Fig. 16b.
- The shearing stress $\tau_{xy}$ can be determined from $\tau_{xy} = \frac{T_c}{J}$.

Let assume that differential element of Fig. 16b has length $dx$, height $dy$, and thickness $dz$.

- If a shearing force $V_x = \tau_{xy} dx dy$ is applied to the top surface of the element, the equation of equilibrium $\sum F_x = 0$ then will require application of an opposite shear force $V'_x$ at the bottom of the element.
Other Stresses Induced By Torsion

\[ V_x = \tau_{yx} dx \; dz \]
\[ V_y = \tau_{xy} dy \; dz \]

\[ \Delta z \]
\[ \Delta y \]
\[ \Delta x \]

![Diagram of stresses in oblique planes](image)

Fig. 17

Other Stresses Induced By Torsion

- If \( \Sigma F_x = 0 \) then requires application of an opposite shear force \( V_x \) at the bottom of the element, then it will the element subjected to a clockwise couple.
- This clockwise couple must be balanced by counterclockwise couple composed of \( V_x \) applied to the vertical faces of the element.
Other Stresses Induced By Torsion

The application of the equilibrium moment equation $\sum M_z = 0$ gives:

$$\tau_{yx}(dx\,dz)\,dy = \tau_{xy}(dy\,dz)\,dx$$

From which the important result:

$$\tau_{yx} = \tau_{xy} \quad (27)$$

Other Stresses Induced By Torsion

If the equations of equilibrium are applied to the free-body diagram of Fig. 16c (which is a wedge-shaped part of the differential element of Fig. 16b with $dA$ being the area of the inclined face), the following results are obtained:

$$\sum F_i = 0$$

$$\tau_n dA - \tau_{xy} (dA \cos \alpha) \cos \alpha + \tau_{yx} (dA \sin \alpha) \sin \alpha = 0 \quad (28)$$
Stresses in Oblique Planes

- Other Stresses Induced By Torsion

\[ \sum F_i = 0 \]
\[ \tau_{yw} dA - \tau_{yx} (dA \cos \alpha) \cos \alpha + \tau_{xy} (dA \sin \alpha) \sin \alpha = 0 \]

From which
\[ \tau_{yw} = \tau_{yx} \left( \cos^2 \alpha - \sin^2 \alpha \right) = \tau_{yx} \cos 2\alpha \quad (29) \]
Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - Likewise, if we take summation of forces in the $n$ direction (see Fig. 16c), then the results would be

\[ + \sum F_n = 0 \]
\[ \sigma_n dA - \tau_{xy} (dA \cos \alpha) \sin \alpha - \tau_{yx} (dA \sin \alpha) \cos \alpha = 0 \quad (30) \]

- Other Stresses Induced By Torsion
  - From which

\[ \sigma_n = 2 \tau_{xy} \sin \alpha \cos \alpha = \tau_{xy} \sin 2\alpha \quad (31) \]

Fig. 16c
Stresses in Oblique Planes

- **Maximum Normal Stress due to Torsion on Circular Shaft**
  
  The maximum compressive normal stress \( \sigma_{\text{max}} \) can be computed from
  \[
  \sigma_{\text{max}} = \tau_{\text{max}} = \frac{T_{\text{max}} c}{J}
  \]  
  (32)

---

Stresses in Oblique Planes

- **Example 4**
  
  A cylindrical tube is fabricated by butt-welding a 6 mm-thick steel plate along a spiral seam as shown. If the maximum compressive stress in the tube must be limited to 80 MPa, determine (a) the maximum torque \( T \) that can be applied and (b) the factor of safety with respect to the failure by fracture for the weld, when a torque of 12 kN.m is applied, if the ultimate strengths of the weld metal are 205 MPa in shear and 345 MPa in tension.
Stresses in Oblique Planes

Example 4 (cont’d)

(a) The polar moment of area for the cylindrical tube can be determined from Eq. 14 as

\[ J = \frac{\pi}{2} \left( r_o^4 - r_i^4 \right) = \frac{\pi}{2} \left( \frac{150}{2} \right)^4 - \left( \frac{150 - 6}{2} \right)^4 = 14.096 \times 10^6 \text{ mm}^4 \]

The maximum torque can be computed from Eq. 32 as

\[ \sigma_{\text{max}} = \frac{T_{\text{max}} c}{J} \Rightarrow T_{\text{max}} = \frac{\sigma_{\text{max}} J}{c} = \frac{80 \times 10^6 (14.096 \times 10^{-6})}{75 \times 10^{-3}} \]

\[ = 15.036 \times 10^3 \text{ N} \cdot \text{m} = 15.036 \text{ kN} \cdot \text{m} \]
Example 4 (cont’d)

(b) The normal stress $\sigma_n$ and shear stress $\tau_{nt}$ on the weld surface are given by Eqs. 30 and 29 as

$$\sigma_n = \tau_{ys} \sin 2\alpha = \frac{T_c}{J} \sin 2\alpha = \frac{12 \times 10^3(75 \times 10^{-3})}{14.096 \times 10^{-6}} \sin 2(60^\circ) = 55.29 \text{ MPa (T)}$$

$$\tau_{ul} = \tau_{ys} \cos 2\alpha = \frac{T_c}{J} \cos 2\alpha = \frac{12 \times 10^3(75 \times 10^{-3})}{14.096 \times 10^{-6}} \cos 2(60^\circ) = -31.92 \text{ MPa}$$

Example 4 (cont’d)

The factors of safety with respect to failure by fracture for the weld are

$$FS_\sigma = \frac{\sigma_{ult}}{\sigma_n} = \frac{345}{55.29} = 6.24$$

$$FS_\tau = \frac{\tau_{ult}}{\tau_{nt}} = \frac{205}{31.92} = 6.42$$
Statically Indeterminate Shafts

- Up to this point, all problems discussed are statically determinate, that is, only the equations of equilibrium were required to determine the torque $T$ at any section of the shaft.
- It is often for torsionally loaded members to be statically indeterminate in real engineering applications.

Statically Indeterminate Shafts

- When this occurs, distortion equations involving angle of twist $\theta$ must written until the total number of equations agrees with the number of unknowns to be determined.
- A simplified angle of twist diagram will often be of great assistance in obtaining the correct equations.
Example 5

A steel shaft and aluminum tube are connected to a fixed support and to a rigid disk as shown in the figure. Knowing that the initial stresses are zero, determine the minimum torque $T_0$ that may be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use $G = 80$ GPa for steel and $G = 27$ GPa for aluminum.
Example 5 (cont’d)

- Free-body diagram for the rigid disk

From statics,
\[ T_0 = T_{al} + T_{st} \quad (39) \]

\[ \theta_{al} = \theta_{st} \Rightarrow \frac{T_{al}L_{al}}{J_{al}G_{al}} = \frac{T_{st}L_{st}}{J_{st}G_{st}} \quad (40) \]

Example 5 (cont’d)

- Properties of the aluminum tube

- Properties of the aluminum tube

\[ G_{al} = 27 \text{ GPa} \]
\[ r_i = 30 \text{ mm} = 0.030 \text{ m} \]
\[ r_o = 38 \text{ mm} = 0.038 \text{ m} \]

\[ J_{al} = \frac{\pi}{2} \left[ (0.038)^4 - (0.030)^4 \right] = 2.003 \times 10^{-6} \text{ m}^4 \]
Example 5 (cont’d)

– Properties of the steel tube

\[ G_{st} = 80 \text{ GPa} \]

\[ c = 25 \text{ mm} = 0.025 \text{ m} \]

\[ J_{st} = \frac{\pi}{2} (0.025)^4 = 0.6136 \times 10^{-6} \text{ m}^4 \]

Substituting these input values in Eq. 40, gives

\[
\frac{T_{al} L_{al}}{J_{al} G_{al}} = \frac{T_{st} L_{st}}{J_{st} G_{st}}
\]

\[
\frac{T_{al} (0.5)}{2.003 \times 10^{-6} (27)} = \frac{T_{st} (0.5)}{0.6136 \times 10^{-6} (80)}
\]

\[
T_{st} = 0.908 T_{al} \quad (41)
\]
Example 5 (cont’d)

Let’s assume that the requirement $\tau_{st}$ is less or to equal to 120 MPa, therefore

$$T_{st} = \frac{\tau_{st}J_{st}}{c_{st}} = \frac{120 \times 10^6 (0.6136 \times 10^{-6})}{0.025} = 2945 \text{ N} \cdot \text{m}$$

From Eq. 39, we have

$$T_{st} = 0.908 T_{al}$$

$$2945 = 0.908 T_{al} \Rightarrow T_{al} = 3244 \text{ N} \cdot \text{m}$$
Example 6

A circular shaft $AB$ consists of a 10-in-long, 7/8 in-diameter steel cylinder, in which a 5-in.-long, 5/8-in.-diameter cavity has been drilled from end $B$. The shaft is attached to fixed supports at both ends, and a 90 lb – ft torque is applied at its mid-section. Determine the torque exerted on the shaft by each of the supports.

Example 6

- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at $A$ and $B$.
- From a free-body analysis of the shaft, \( T_A + T_B = 90\text{ lb} \cdot \text{ft} \)
  which is not sufficient to find the end torques. The problem is statically indeterminate.
- Divide the shaft into two components which must have compatible deformations,
  \[
  \phi = \phi_1 + \phi_2 = \frac{T_A l_1}{J_1 G} - \frac{T_B l_2}{J_2 G} = 0
  \quad T_B = \frac{l_2}{l_2 + l_1} T_A
  \]
- Substitute into the original equilibrium equation,
  \[
  T_A + \frac{l_2}{l_2 + l_1} T_A = 90\text{ lb} \cdot \text{ft}
  \]