LECTURE

6

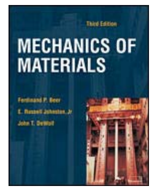
Chapter 3.1 - 3.5

The McGraw·Hill Companies

Third Edition

SHAFTS: TORSION LOADING AND DEFORMATION

A. J. Clark School of Engineering •Department of Civil and Environmental Engineering



by

Dr. Ibrahim A. Assakkaf

SPRING 2003

ENES 220 – Mechanics of Materials

Department of Civil and Environmental Engineering

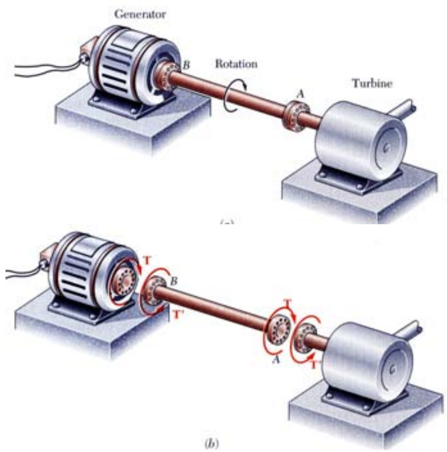
University of Maryland, College Park

Torsion Loading

- Introduction
  - – Members subjected to axial loads were discussed previously.
  - – The procedure for deriving load-deformation relationship for axially loaded members was also illustrated.
  - – This chapter will present a similar treatment of members subjected to torsion by loads that to twist the members about their longitudinal centroidal axes.

Torsion Loading

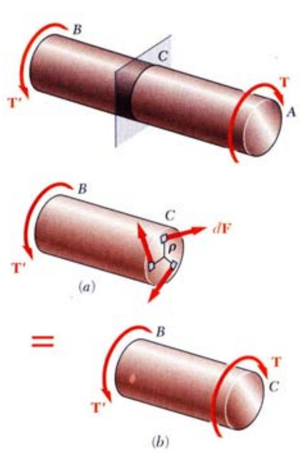
Torsional Loads on Circular Shafts



- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T*'

Torsion Loading

Net Torque Due to Internal Stresses



- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho(\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

Torsion Loading

- Introduction

Cylindrical members

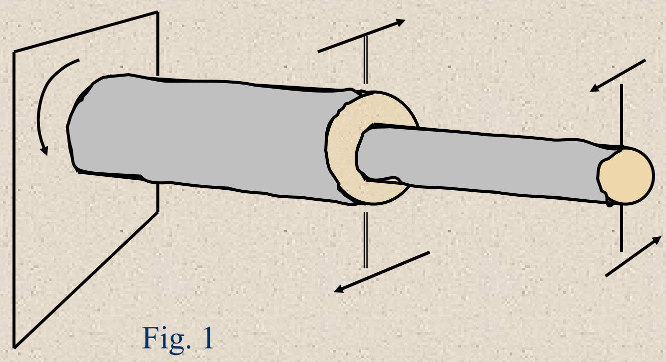


Fig. 1

Torsion Loading

- Introduction

Rectangular members

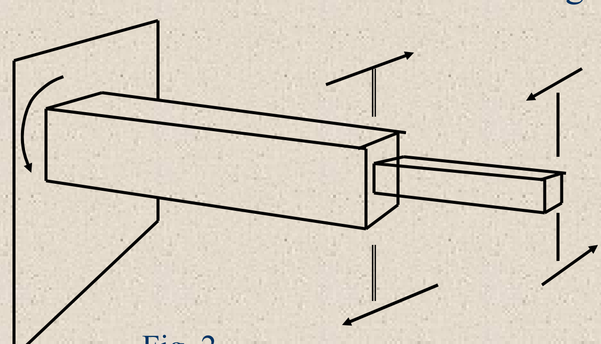


Fig. 2

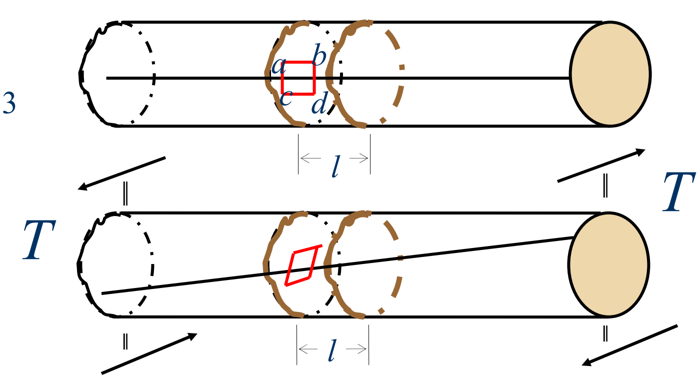
Torsion Loading

- Introduction
  - This chapter deals with members in the form of concentric circular cylinders, solid and hollow, subjected to torques about their longitudinal geometric axes.
  - Although this may seem like a somewhat a special case, it is evident that many torque-carrying engineering members are cylindrical in shape.

Torsion Loading

- Deformation of Circular Shaft
  - Consider the following shaft

Fig. 3



Torsion Loading

- Deformation of Circular Shaft
  - In reference to the previous figure, the following observations can be noted:
    - The distance l between the outside circumferential lines does not change significantly as a result of the application of the torque. However, the rectangles become parallelograms whose sides have the same length as those of the original rectangles.

Torsion Loading

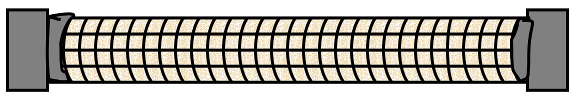
- The circumferential lines do not become zigzag; that is ; they remain in parallel planes.
- The original straight parallel longitudinal lines, such as ab and cd, remain parallel to each other but do not remain parallel to the longitudinal axis of the member. These lines become helices*.

*A helix is a path of a point that moves longitudinally and circumferentially along a surface of a cylinder at uniform rate

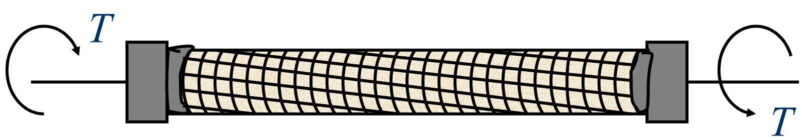
Torsion Loading

- Deformation of Circular Shaft Subjected to Torque T

Fig. 4



a

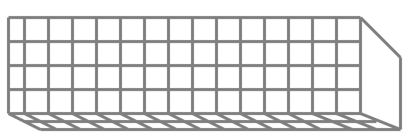


b

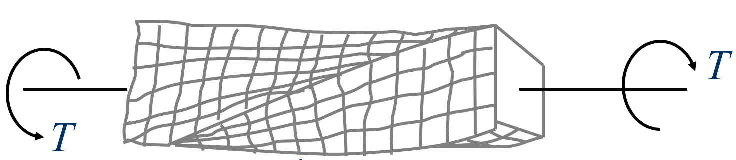
Torsion Loading

- Deformation of a Bar of Square Cross Section Subjected to Torque T

Fig. 5



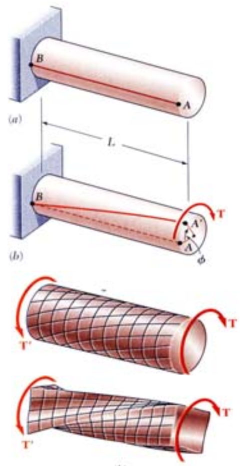
a



b

Torsion Loading

Shaft Deformations



- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

  $\phi \propto T$

  $\phi \propto L$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Torsion Loading

- An Important Property of Circular Shaft
  - When a circular shaft is subjected to torsion, every cross section remains plane and disturbed
  - In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab.

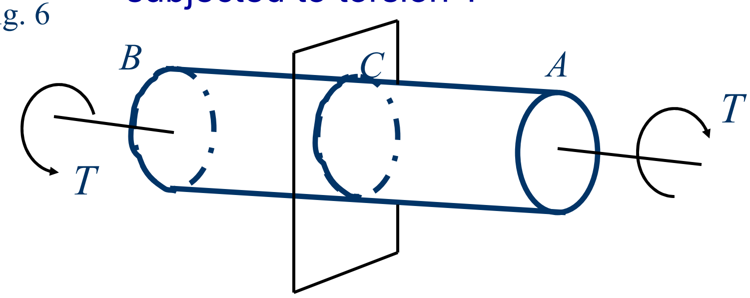
Torsion Loading

- An Important Property of Circular Shaft
  - This illustrated in Fig.4b, which shows the deformation in rubber model subjected to torsion.
  - This property applies to circular shafts whether solid or hollow.
  - It does not apply to noncircular cross section. When a bar of square cross section is subjected to torsion, its various sections are warped and do not remain plane (see Fig. 5.b)

Torsion Loading

- Stresses in Circular Shaft due to Torsion
  - Consider the following circular shaft that is subjected to torsion T

Fig. 6



Torsion Loading

- Stresses in Circular Shaft due to Torsion
  - A section perpendicular to the axis of the shaft can be passed at an arbitrary point C as shown in Fig. 6.
  - The Free-body diagram of the portion BC of the shaft must include the elementary shearing forces dF perpendicular to the radius ρ of the shaft.

Torsion Loading

- Stresses in Circular Shaft due to Torsion

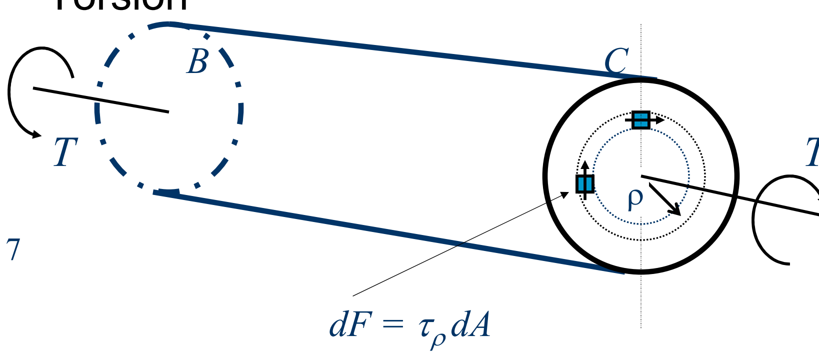


Fig. 7

Torsion Loading

- Stresses in Circular Shaft due to Torsion
  - But the conditions of equilibrium for *BC* require that the system of these elementary forces be equivalent to an internal torque *T.*
  - Denoting ρ the perpendicular distance from the force d*F* to axis of the shaft, and expressing that the sum of moments of

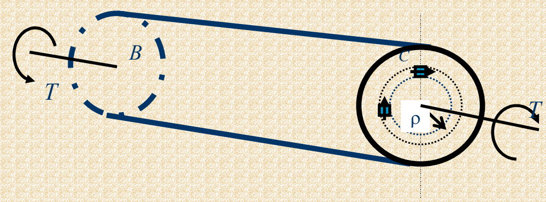
Torsion Loading

- Stresses in Circular Shaft due to Torsion
  - of the shearing forces d*F* about the axis of the shaft is equal in magnitude to the torque *T*, we can write

$$T = T_r = \int_{\text{area}} \rho \, dF = \int_{\text{area}} \rho \, \tau \, \mathrm{d}A \qquad (1)$$

Torsion Loading

- Stresses in Circular Shaft due to Torsion



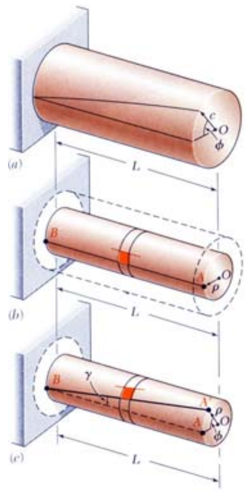
$$T = T_r = \int_{\text{area}} \rho\, \tau\, \mathrm{d}A \qquad (2)$$

Torsional Shearing Strain

- If a plane transverse section before twisting remains plane after twisting and a diameter of the the section remains straight, the distortion of the shaft of Figure 7 will be as shown in the following figures (Figs. 8 and 9):

Torsional Shearing Strain

Shearing Strain



- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shear strain is proportional to twist and radius

$$\gamma_{max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{max}$$

Torsional Shearing Strain

- Shearing Strain

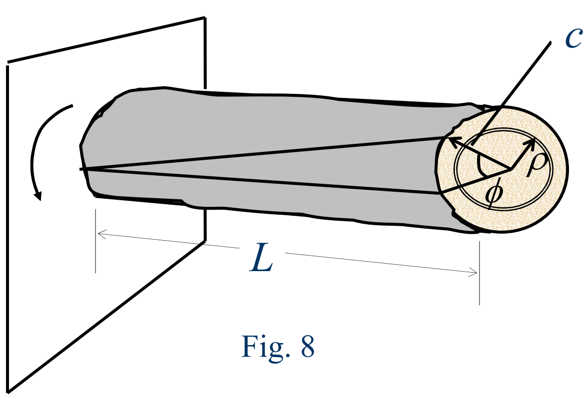


Fig. 8

Torsional Shearing Strain

- Shearing Strain

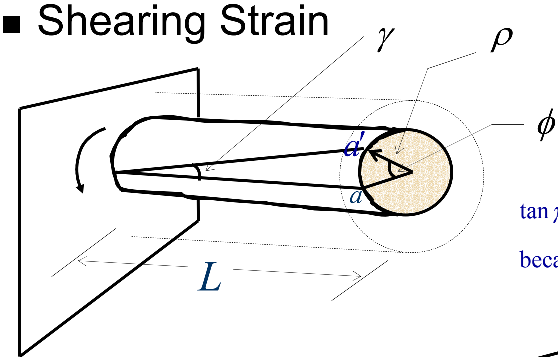
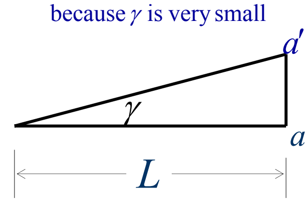


Fig. 9

$$\tan\gamma = \frac{aa'}{L} \approx \sin\frac{aa'}{L}$$

because γ is very small



Torsional Shearing Strain

- Shearing Strain
  - From Fig. 9, the length aá can be expressed as

$$aa' = L\tan\gamma = L\gamma \qquad (3)$$

  - But

$$aa' = \rho\phi \qquad (4)$$

  - Therefore,

$$L\gamma = \rho\phi \Rightarrow \gamma = \frac{\rho\phi}{L} \qquad (5)$$

Torsional Shearing Strain

- Shearing Strain

For radius ρ, the shearing strain for circular shaft is

$$\gamma_\rho = \frac{\rho\phi}{L} \qquad (6)$$

For radius c, the shearing strain for circular shaft is

$$\gamma_c = \frac{c\phi}{L} \qquad (7)$$

Torsional Shearing Strain

- Shearing Strain

Combining Eqs. 6 and 7, gives

$$\phi = \frac{\gamma_\rho L}{\rho} = \frac{\gamma_c L}{c} \qquad (8)$$

Therefore

$$\gamma_\rho = \frac{\gamma_c}{c}\rho \qquad (9)$$

Torsional Shearing Stress

- The Elastic Torsion Formula

  If Hooke's law applies, the shearing stress τ is related to the shearing strain γ by the equation

$$\tau = G\gamma \qquad (10)$$

  where G = modulus of rigidity. Combining Eqs. 9 and 10, results in

$$\frac{\tau_\rho}{G} = \frac{\tau_c}{Gc}\rho \Rightarrow \tau_\rho = \frac{\tau_c}{c}\rho \qquad (11)$$

Torsional Shearing Stress

- The Elastic Torsion Formula

  When Eq. 11 is substituted into Eq. 2, the results will be as follows:

$$T = T_r = \int_{\text{area}} \rho \, \tau \, \mathrm{d}A$$

$$= \int_0^c \rho \left(\frac{\tau_c}{c} \rho \right) \mathrm{d}A = \int_{\text{area}} \rho \left(\frac{\tau_c}{c} \rho \right) \mathrm{d}A$$

$$= \frac{\tau_c}{c} \int_{\text{area}} \rho^2 \mathrm{d}A \qquad (12)$$

Torsional Shearing Stress

- Polar Moment of Inertia

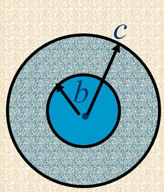
  The integral of equation 12 is called the polar moment of inertia (polar second moment of area).

  It is given the symbol *J*. For a solid circular shaft, the polar moment of inertia is given by

$$J = \int \rho^2 \mathrm{d}A = \int_0^c \rho^2 \left(2\pi\rho \, \mathrm{d}\rho\right) = \frac{\pi c^4}{2} \qquad (13)$$

Torsional Shearing Stress

- Polar Moment of Inertia
  - For a circular annulus as shown, the polar moment of inertia is given by



$$J = \int \rho^2 \mathrm{d}A = \int_b^c \rho^2 \left(2\pi\rho \, \mathrm{d}\rho\right)$$

$$= \frac{\pi c^4}{2} - \frac{\pi b^4}{2} = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) \qquad (14)$$

r_0 = outer radius and r_i = inner radius

Torsional Shearing Stress

- Shearing Stress in Terms of Torque and Polar Moment of Inertia
  - In terms of the polar second moment *J*, Eq. 12 can be written as

$$T = T_r = \frac{\tau_c}{c} \int_{\text{area}} \rho^2 \mathrm{d}A = \frac{\tau_c J}{c} \qquad (15)$$

  - Solving for shearing stress,

$$\tau_c = \frac{Tc}{J} \qquad (16)$$

Torsional Shearing Stress

- Shearing Stress in Terms of Torque and Polar Moment of Inertia

$$\tau_{\max} = \frac{Tc}{J} \qquad (17a)$$

$$\tau_\rho = \frac{T\rho}{J} \qquad (18a)$$

τ= shearing stress, T = applied torque
ρ = radius, and J = polar moment on inertia

Torsional Shearing Stress

- Distribution of Shearing Stress within the Circular Cross Section

$$\tau_c = \frac{Tc}{J}$$

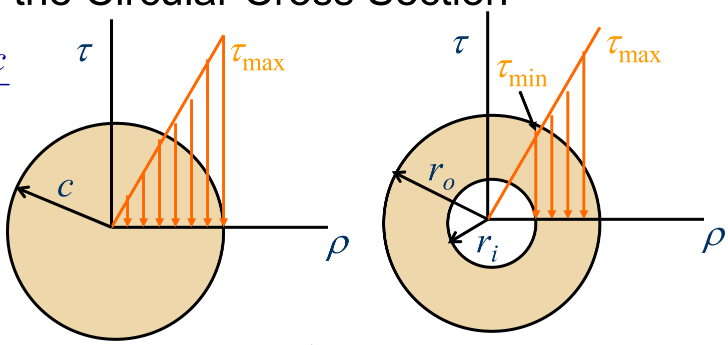
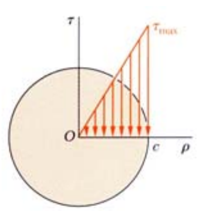
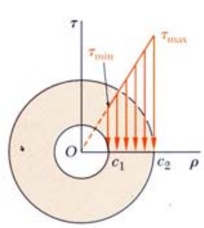


Fig. 10

Stresses in Elastic Range



$J = \frac{1}{2}\pi c^4$



$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$

- Multiplying the previous equation by the shear modulus,

  $$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

  From Hooke's Law, $\tau = G\gamma$, so

  $$\tau = \frac{\rho}{c}\tau_{max}$$

  The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

  $$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

- The results are known as the elastic torsion formulas,

  $$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Torsional Shearing Stress

- Example 1
  - A hollow cylindrical steel shaft is 1.5 m long and has inner and outer diameters equal to 40 mm and 60 mm. (a) What is the largest torque which may be applied to the shaft if the shearing stress is not to exceed 120 MPa? (b) What is the corresponding minimum value of the shearing stress in the shaft?

Torsional Shearing Stress

- Example 1 (cont'd)

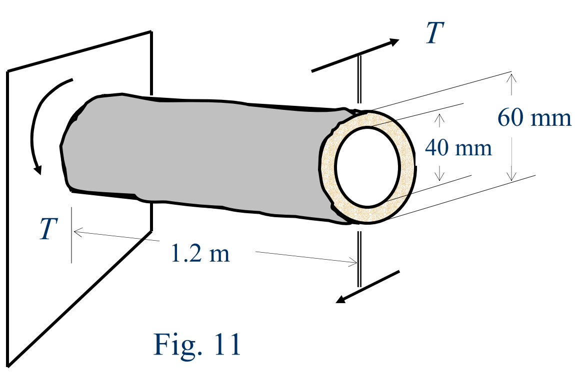


Fig. 11

Torsional Shearing Stress

- Example 1 (cont'd)
  - (a) Largest Permissible Torque

Using Eq.17a

$$\tau_{\max} = \frac{Tc}{J} \Rightarrow T_{\max} = \frac{\tau_{\max} J}{c} \qquad (19)$$

Using Eq.14 for claculating J,

$$J = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) = \frac{\pi}{2}\left[(0.03)^4 - (0.02)^4\right]$$
$$= 1.021 \times 10^{-6} \text{ m}^4$$

Torsional Shearing Stress

- Example 1 (cont'd)

Substituting for J and τ_{max} into Eq. 19, we have

$$T = \frac{J\tau_{\max}}{r_o} = \frac{\left(1.021 \times 10^{-6}\right)\left(120 \times 10^6\right)}{0.03} = 4.05 \text{ kN} \cdot \text{m}$$

(b) Minimum Shearing Stress

$$\tau_\rho = \frac{Tr_i}{J} = \frac{4.05 \times 10^3 (0.02)}{1.021 \times 10^{-6}} = 79.3 \text{ MPa}$$

Torsional Displacements

- Angle of Twist in the Elastic Range
  - Often, the amount of twist in a shaft is of importance.
  - Therefore, determination of angle of twist is a common problem for the machine designer.
  - The fundamental equations that govern the amount of twist were discussed previously

Torsional Displacements

- Angle of Twist in the Elastic Range
  - The basic equations that govern angle of twist are
  - Recall Eqs. 6,

$$\gamma_\rho = \frac{\rho\phi}{L} \quad \text{or} \quad \gamma_\rho = \rho\frac{d\theta}{dL} \qquad (20)$$

Torsional Displacements

- Angle of Twist in the Elastic Range

$$\tau_c = \frac{Tc}{J} \quad \text{or} \quad \tau_\rho = \frac{T\rho}{J} \qquad (21)$$

and

$$G = \frac{\tau}{\gamma} \qquad (22)$$

Torsional Displacements

- Angle of Twist in the Elastic Range
  - Recall Eq. 17a and 7

$$\tau_{\max} = \frac{Tc}{J} \qquad \gamma_{\max} = \frac{c\theta}{L}$$

  - Combining these two equations, gives

$$\theta = \frac{\gamma_{\max} L}{c} = \frac{\tau_{\max}}{G}\frac{L}{c} = \left(\frac{Tc}{J}\right)\frac{1}{G}\frac{L}{c}$$

$$= \frac{TL}{GJ}$$

Torsional Displacements

- Angle of Twist in the Elastic Range

  The angle of twist for a circular uniform shaft subjected to external torque *T* is given by

$$\theta = \frac{TL}{GJ} \qquad (22)$$

Torsional Displacements

- Angle of Twist in the Elastic Range
  - Multiple Torques/Sizes
    - The expression for the angle of twist of the previous equation may be used only if the shaft is homogeneous (constant G) and has a uniform cross sectional area A, and is loaded at its ends.
    - If the shaft is loaded at other points, or if it consists of several portions of various cross sections, and materials, then

Torsional Displacements

- Angle of Twist in the Elastic Range
  - Multiple Torques/Sizes
    - It needs to be divided into components which satisfy individually the required conditions for application of the formula.
    - Denoting respectively by T_i, L_i, J_i, and G_i, the internal torque, length, polar moment of area, and modulus of rigidity corresponding to component i, then

$$\theta = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i} \qquad (23)$$

Torsional Displacements

- Multiple Torques/Sizes $\theta = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$

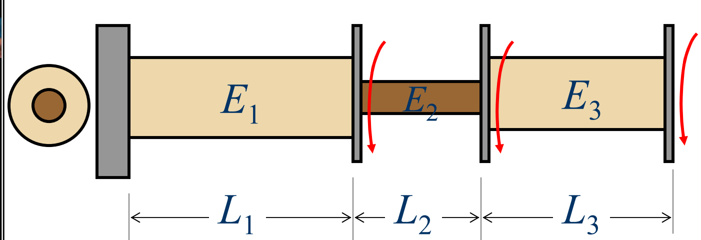


Fig. 12 Circular Shafts

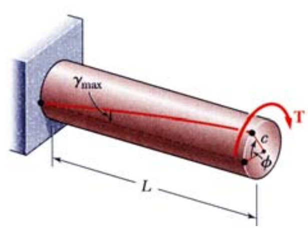
Torsional Displacements

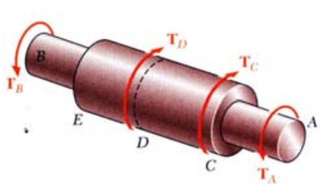
- Angle of Twist in the Elastic Range

  The angle of twist of various parts of a shaft of uniform member can be given by

$$\theta = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i} \qquad (24)$$

Angle of Twist in Elastic Range





- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Torsional Displacements

- Angle of Twist in the Elastic Range

  If the properties (*T, G,* or *J*) of the shaft are functions of the length of the shaft, then

$$\theta = \int_0^L \frac{T}{GJ}\,dx \qquad (25)$$

Torsional Displacements

- Angle of Twist in the Elastic Range
  - Varying Properties

$$\theta = \int_0^L \frac{T}{GJ}\,dx$$

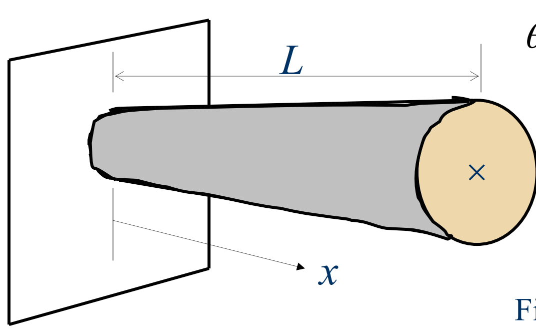


Fig. 13

Torsional Displacements

- Example 2
  - What torque should be applied to the end of the shaft of Example 1 to produce a twist of 2^0? Use the value G = 80 GPa for the modulus of rigidity of steel.

Torsional Displacements

- Example 2 (cont'd)

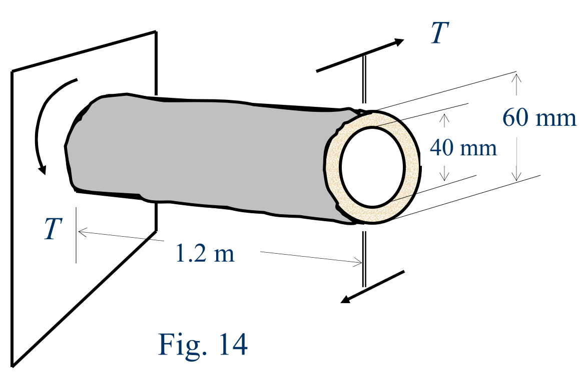


Fig. 14

Torsional Displacements

- Example 2 (cont'd)

  Solving Eq. 22 for *T*, we get

$$T = \frac{JG}{L}\theta \qquad (26)$$

Substituting the given values

$$G = 80 \times 10^9 \text{ Pa} \qquad L = 1.5 \text{ m}$$

$$\theta = 2^0\left(\frac{2\pi \text{ rad}}{360^0}\right) = 34.9 \times 10^{-3} \text{ rad}$$

Torsional Displacements

- Example 2 (cont'd)

  From Example 1, *J* was computed to give a value of 1.021×10^{-6} m^4.

  Therefore,using Eq. 26

$$T = \frac{JG}{L}\theta = \frac{\left(1.021 \times 10^{-6}\right)\left(80 \times 10^9\right)}{1.5}\left(34.9 \times 10^{-3}\right)$$

$$= 1.9 \times 10^3 \text{ N} \cdot \text{m} = 1.9 \text{ kN} \cdot \text{m}$$

Torsional Displacements

- Example 3

  What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft of Examples 1 and 2?

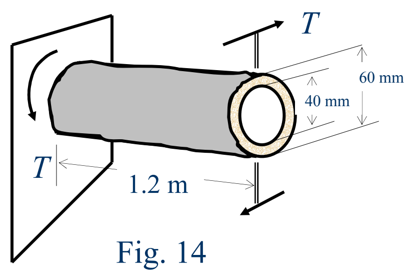


Fig. 14

Torsional Displacements

- Example 3 (cont'd)

$$\tau_\rho = \frac{T\rho}{J} \Rightarrow T = \frac{J\tau_\rho}{\rho}$$

$$= \frac{(1.021\times10^{-6})(70\times10^{6})}{0.02} = 3.5735\ \text{kN}\cdot\text{m}$$

$$\phi = \frac{TL}{GJ} = \frac{3.5735\times10^{3}(1.5)}{80\times10^{9}(1.021\times10^{-6})} = 0.65625$$

To obtain θ in degrees, we write

$$\theta = 0.65625\frac{360}{2\pi} = 3.76^{0}$$