LECTURE

3

Chapter 2.8

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RODS: AXIAL LOADING AND DEFORMATION

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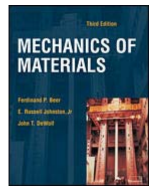

by
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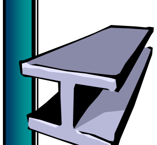
ENES 220 – Mechanics of Materials

Department of Civil and Environmental Engineering
University of Maryland, College Park

Deformations of Members under Axial Loading

- Uniform Member
  – Consider a homogeneous rod as shown in the figure of the next viewgraph.
  – If the resultant axial stress σ = P/A dos not exceed the proportional limit of the material, then Hooke's law can applied, that is



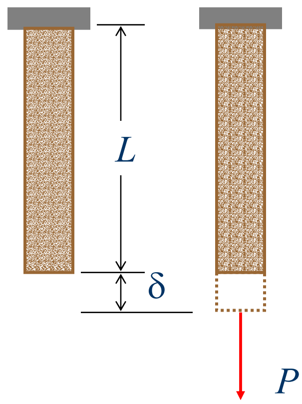
$$\sigma = E\varepsilon$$

σ = axial stress, ε = axial strain

E = modulus of elasticity

Deformations of Members under Axial Loading

- Uniform Member



Deformations of Members under Axial Loading

- Uniform Member
  - From Hooke’s law, it follows that

$$\sigma = E\varepsilon \quad \text{(Hooke's law)}$$

$$\frac{P}{A} = E\varepsilon$$

$$\text{But } \varepsilon = \frac{\delta}{L}, \text{ therefore,}$$

$$\frac{P}{A} = E\frac{\delta}{L} \Rightarrow \delta = \frac{PL}{EA}$$

Deformations of Members under Axial Loading

- Uniform Member
  - The deflection (deformation), δ, of the uniform member subjected to axial loading P is given by

$$\delta = \frac{PL}{EA} \qquad (1)$$

Deformations of Members under Axial Loading

- Multiple Loads/Sizes
  - The expression for the deflection of the previous equation may be used only if the rod or the member is homogeneous (constant E) and has a uniform cross sectional area A, and is loaded at its ends.
  - If the member is loaded at other points, or if it consists of several portions of various cross sections, and materials, then

Deformations of Members under Axial Loading

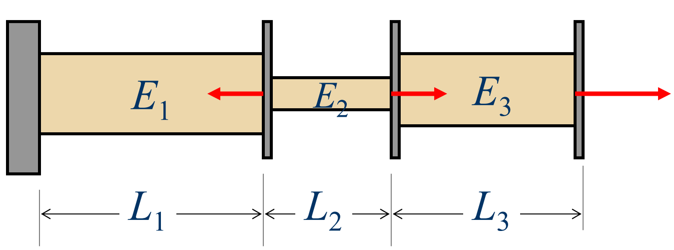
- Multiple Loads/Sizes
  - It needs to be divided into components which satisfy individually the required conditions for application of the formula.
  - Denoting respectively by Pi, Li, Ai, and Ei, the internal force, length, cross-sectional area, and modulus of elasticity corresponding to component i,then

$$\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}$$

Deformations of Members under Axial Loading

- Multiple Loads/Sizes

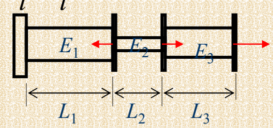
$$\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}$$



Deformations of Members under Axial Loading

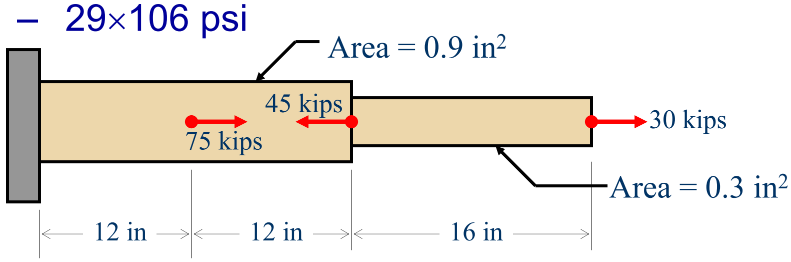
- Multiple Loads/Sizes
  - The deformation of of various parts of a rod or uniform member can be given by

$$\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i} \qquad (2)$$



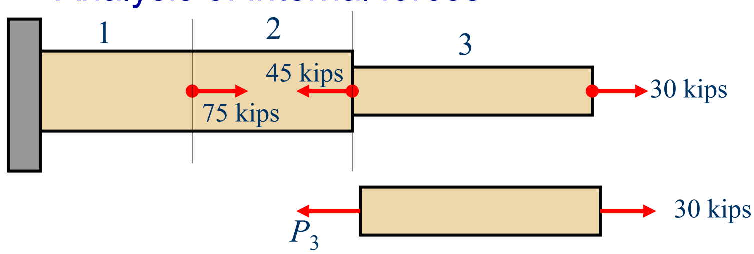
Deformations of Members under Axial Loading

- Example 4
  - Determine the deformation of the steel rod shown under the given loads. Assume that the modulus of elasticity for all parts is
  - 29×106 psi



Deformations of Members under Axial Loading

- Example 4 (cont'd)
  - Analysis of internal forces

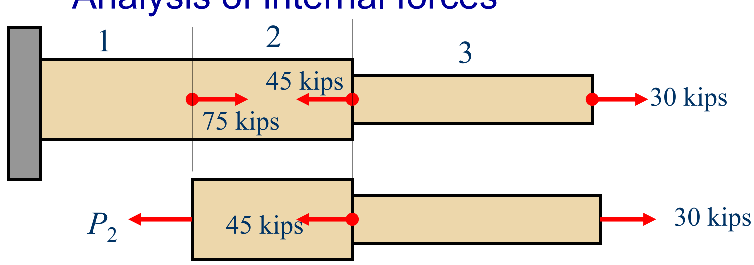


$$\rightarrow + \sum F_x = 0; -P_3 + 30 = 0$$

$$\therefore P_3 = 30 \text{ kips}$$

Deformations of Members under Axial Loading

- Example 4 (cont'd)
  - Analysis of internal forces

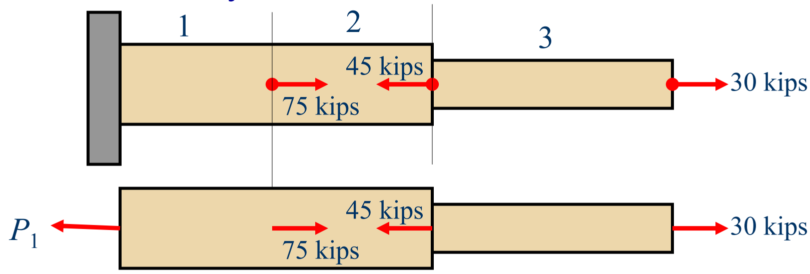


$$\rightarrow + \sum F_x = 0; -P_2 - 45 + 30 = 0$$

$$\therefore P_2 = -15 \text{ kips}$$

Deformations of Members under Axial Loading

- Example 4 (cont'd)
  - Analysis of internal forces



$$\rightarrow + \sum F_x = 0; -P_1 + 75 - 45 + 30 = 0$$

$$\therefore P_1 = 60 \text{ kips}$$

Deformations of Members under Axial Loading

- Example 4 (cont'd)
  - Deflection
    - Input parameters

$L_1 = 12 \text{ in}$ $\quad L_2 = 12 \text{ in}$ $\quad L_3 = 16 \text{ in}$

$A_1 = 0.9 \text{ in}^2$ $\quad A_2 = 0.9 \text{ in}^2$ $\quad A_3 = 0.3 \text{ in}^2$

From analysis of internal forces,

$P_1 = 60 \text{ kips} = 60{,}000 \text{ lb}$

$P_2 = -15 \text{ kips} = -15{,}000 \text{ lb}$

$P_3 = 30 \text{ kips} = 30{,}000 \text{ lb}$

Deformations of Members under Axial Loading

- Example 4 (cont'd)
  - Carrying the values into the deformation formula:

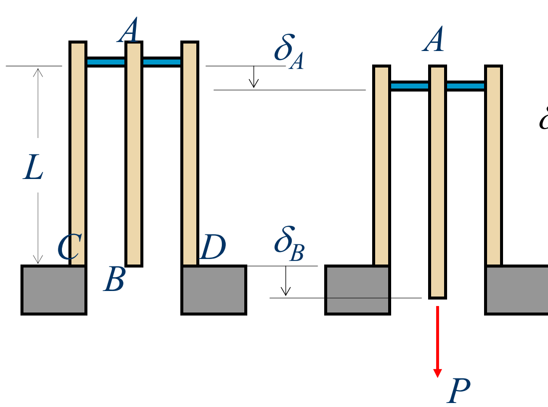
$$\delta = \sum_{i=1}^{3} \frac{P_i L_i}{E_i A_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3} \right)$$

$$\frac{1}{29\times 10^6}\left(\frac{60{,}000(12)}{0.9} + \frac{-15{,}000(12)}{0.9} + \frac{30{,}000(16)}{0.3} \right)$$

$$\delta = 0.0759 \text{ in}$$

Deformations of Members under Axial Loading

- Relative Deformation



$$\delta_{B/A} = \delta_B - \delta_A = \frac{PL}{AE}$$

Deformations of Members under Axial Loading

- Relative Deformation
  - If the load P is applied at B, each of the three bars will deform.
  - Since the bars AC and AD are attached to the fixed supports at C and D, their common deformation is measured by the displacement δA at point A.

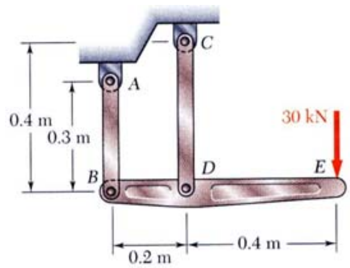
Deformations of Members under Axial Loading

- Relative Deformation
  - On the other hand, since both ends of bars *AB* move, the deformation of *AB* is measured by the difference between the displacements δ_A and δ_B of points *A* and *B*.
  - That is by ***relative displacement*** of *B* with respect to *A*, or

$$\delta_{B/A} = \delta_B - \delta_A = \frac{PL}{EA} \quad (3)$$

Deformations of Members under Axial Loading

- Example 5



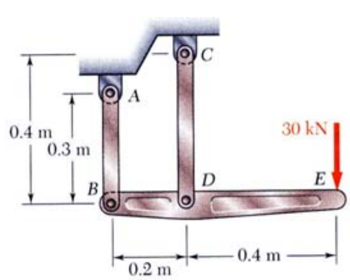
The rigid bar *BDE* is supported by two links *AB* and *CD*.

Link *AB* is made of aluminum (E = 70 GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel (E = 200 GPa) and has a cross-sectional area of (600 mm^2).

For the 30-kN force shown, determine the deflection a) of *B*, b) of *D*, and c) of *E*.

Deformations of Members under Axial Loading

- Example 5 (cont'd)

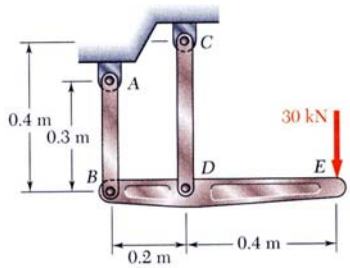


SOLUTION:

- Apply a free-body analysis to the bar *BDE* to find the forces exerted by links *AB* and *DC*.
- Evaluate the deformation of links *AB* and *DC* or the displacements of *B* and *D*.
- Work out the geometry to find the deflection at E given the deflections at B and D.

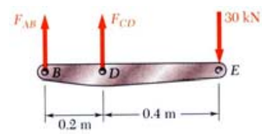
Deformations of Members under Axial Loading

- Example 5 (cont'd)



SOLUTION:

Free body: Bar *BDE*



$$\sum M_B = 0$$

$$0 = -(30\,\text{kN} \times 0.6\,\text{m}) + F_{CD} \times 0.2\,\text{m}$$

$$F_{CD} = +90\,\text{kN} \quad \textit{tension}$$

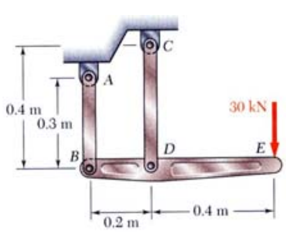
$$\sum \text{M}_\text{D} = 0$$

$$0 = -(30\,\text{kN} \times 0.4\,\text{m}) - F_{AB} \times 0.2\,\text{m}$$

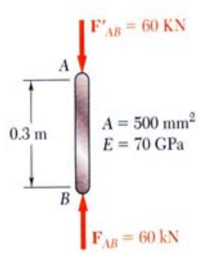
$$F_{AB} = -60\,\text{kN} \quad \textit{compression}$$

Deformations of Members under Axial Loading

- Example 5 (cont'd)



Displacement of *B*:



$$\delta_B = \frac{PL}{AE}$$

$$= \frac{(-60\times10^3\,\text{N})(0.3\,\text{m})}{(500\times10^{-6}\,\text{m}^2)(70\times10^9\,\text{Pa})}$$

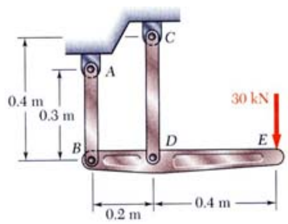
$$= -514\times10^{-6}\,\text{m}$$

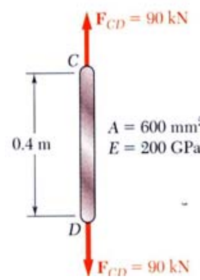
$$\boxed{\delta_B = 0.514\,\text{mm}\uparrow}$$

Deformations of Members under Axial Loading

- Example 5 (cont'd)

Displacement of *D*:





$$\delta_D = \frac{PL}{AE}$$

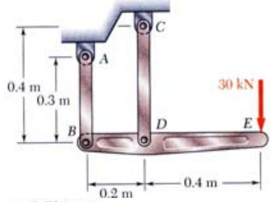
$$= \frac{(90\times10^3\,\text{N})(0.4\,\text{m})}{(600\times10^{-6}\,\text{m}^2)(200\times10^9\,\text{Pa})}$$

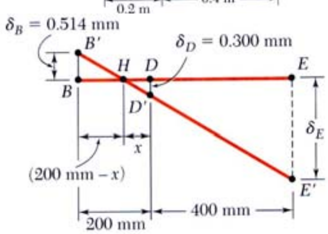
$$= 300\times10^{-6}\,\text{m}$$

$$\boxed{\delta_D = 0.300\,\text{mm}\downarrow}$$

Deformations of Members under Axial Loading

- Example 5 (cont'd)





Displacement of D:

$$\frac{BB'}{DD'} = \frac{BH}{HD}$$

$$\frac{0.514\,\text{mm}}{0.300\,\text{mm}} = \frac{(200\,\text{mm}) - x}{x}$$

$$x = 73.7\,\text{mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD}$$

$$\frac{\delta_E}{0.300\,\text{mm}} = \frac{(400 + 73.7)\text{mm}}{73.7\,\text{mm}}$$

$$\delta_E = 1.928\,\text{mm}$$

$$\boxed{\delta_E = 1.928\,\text{mm}\downarrow}$$

Deformations of Members under Axial Loading

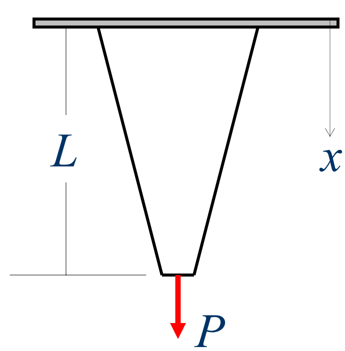
- Nonuniform Deformation
  - For cases in which the axial force or the cross-sectional area varies continuously along the length of the bar, then Eq. 1
  - (PL / EA) is not valid.
  - Recall that in the case of variable cross section, the strain depends on the position of point Q, where it is computed from

$$\varepsilon = \frac{d\delta}{dx}$$

Deformations of Members under Axial Loading

- Nonuniform Deformation

$$\varepsilon = \frac{d\delta}{dx}$$



Deformations of Members under Axial Loading

- Nonuniform Deformation
  - Solving for $d\delta$ and substituting for ε

$$d\delta = \varepsilon dx$$

  - But $\varepsilon = \sigma / E$, and $\sigma = P/A$. therefore

$$d\delta = \varepsilon dx = \frac{\sigma}{E} dx = \frac{P}{EA} dx \qquad (4)$$

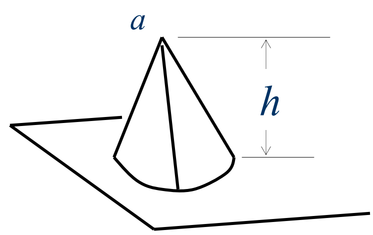
Deformations of Members under Axial Loading

- Nonuniform Deformation
  - The total deformation δ of the rod or bar is obtained by integrating Eq. 4 over the length L as

$$\delta = \int_0^L \frac{P_x}{EA_x} dx \qquad (5)$$

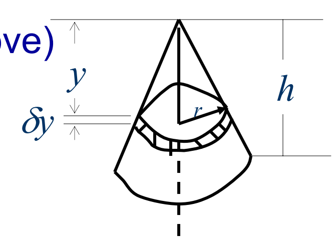
Deformations of Members under Axial Loading

- Example 6
  - Determine the deflection of point a of a homogeneous circular cone of height h, density ρ, and modulus of elasticity E due to its own weight.



Deformations of Members under Axial Loading

- Example 6 (cont'd)
  - Consider a slice of thickness dy
  - P = weight of above slice
  - = ρg (volume above)



$$P = \rho g\left(\frac{1}{3}\pi r^2 y\right)$$

$$d\delta = \frac{P\,dy}{EA} = \frac{\rho g\left(\frac{1}{3}\pi r^2 y\right)}{E\left(\pi r^2\right)} = \frac{\rho g}{3E}\,y\,dy$$

Deformations of Members under Axial Loading

- Example 6 (cont'd)

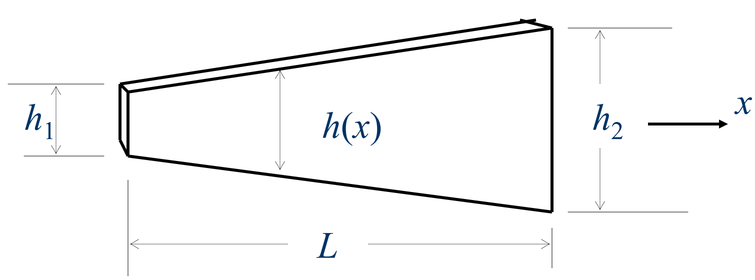
$$d\delta = \frac{\rho g}{3E} y dy$$

$$\delta = \int_0^h \frac{\rho g}{3E} y dy = \frac{\rho g}{3E} \frac{y^2}{2}\Bigg|_0^h$$

$$\delta = \frac{\rho g h^2}{6E} \downarrow$$

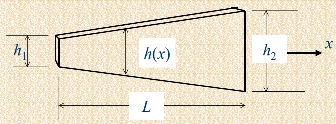
Deformations of Members under Axial Loading

- Normal Stresses in Tapered Bar
  - Consider the following tapered bar with a thickness t that is constant along the entire length of the bar.



Deformations of Members under Axial Loading

- Normal Stresses in Tapered Bar

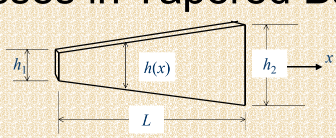


  – The following relationship gives the height *h* of tapered bar as a function of the location *x*

$$h_x = h_1 + (h_2 - h_1)\frac{x}{L} \qquad (6)$$

Deformations of Members under Axial Loading

- Normal Stresses in Tapered Bar

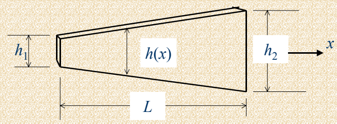


  – The area A_x at any location *x* along the length of the bar is given by

$$A_x = th_x = t\left[h_1 + (h_2 - h_1)\frac{x}{L}\right] \qquad (7)$$

Deformations of Members under Axial Loading

- Normal Stresses in Tapered Bar

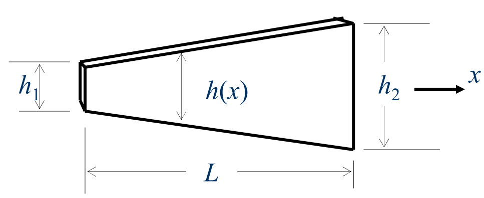


- The normal stress σ_x as a function of x is given

$$\sigma_x = \frac{P}{A_x} = \frac{P}{t\left[h_1 + (h_2 - h_1)\frac{x}{L}\right]} \quad (8)$$

Deformations of Members under Axial Loading

- Example 7
  - Determine the normal stress as a function of x along the length of the tapered bar shown if
  - h1 = 2 in
  - h2 = 6 in
  - t = 3 in, and
  - L = 36 in
  - P = 5,000 lb



Deformations of Members under Axial Loading

- Example 7 (cont'd)
  - Applying Eq. 8, the normal stress as a function of x is given by

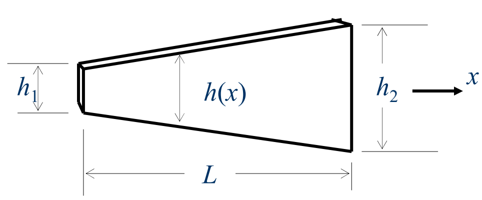
$$\sigma_x = \frac{P}{A_x} = \frac{P}{t\left[h_1 + (h_2 - h_1)\dfrac{x}{L}\right]}$$

$$\sigma_x = \frac{5000}{3\left[2 + (6-2)\dfrac{x}{36}\right]} = \frac{5000}{6 + \dfrac{x}{3}}$$

$$\sigma_x = \frac{15000}{18 + x}$$

Deformations of Members under Axial Loading

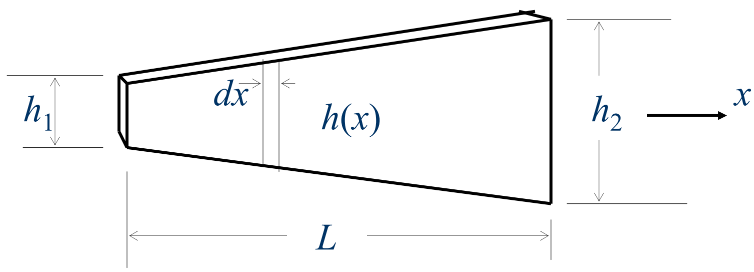
- Example 7 (cont'd)
  - Max σ = 833.3 psi
    - At x =0
  - Min σ = 277.8 psi
    - At x =36 in



x (in)	σ (psi)
0	833.3
3	714.3
6	625.0
9	555.6
12	500.0
15	454.5
18	416.7
21	384.6
24	357.1
27	333.3
30	312.5
33	294.1
36	277.8

Deformations of Members under Axial Loading

- Deflection of Tapered Bar
  - Consider the following tapered bar with a thickness t that is constant along the entire length of the bar.



Deformations of Members under Axial Loading

- Deflection of Tapered Bar
  - Recall Eq. 5

$$\delta = \int_0^L \frac{P_x}{EA_x} dx$$

  - Substitute Eq. 7

$$A_x = t h_x = t\left[h_1 + (h_2 - h_1)\frac{x}{L}\right]$$

  - into Eq. 5, therefore

$$\delta = \frac{PL}{Et}\int_0^L \frac{1}{h_1 L + (h_2 - h_1)x} dx \qquad (9)$$

Deformations of Members under Axial Loading

- Deflection of Tapered Bar

Integrating Eq. 9, the deflection of a tapered bar is given by

$$\delta = \frac{PL}{Et}\left(\frac{1}{h_2 - h_1}\right)\ln\left[\left(h_2 - h_1\right)L\right] \quad (10)$$