ADVANCED TOPICS IN MECHANICS-I

1. Buckling: Eccentric Loading

- References:
ADVANCED TOPICS IN MECHANICS-I

2. Torsion of Noncircular Members and Thin-Walled Hollow Shafts

- References

ADVANCED TOPICS IN MECHANICS-I

3. Introduction to Plastic Moment

- References:
Buckling: Eccentric Loading

- Introduction
  - The Euler formula that was developed earlier was based on the assumption that the concentrated compressive load \( P \) on the column acts though the centroid of the cross section of the column (Fig. 1).
  - In many realistic situations, however, this is not the case. The load \( P \) applied to a column is never perfectly centric.

---

Figure 1. Centric Loading
Buckling: Eccentric Loading

Introduction

- In many structures and buildings, compressive members that used as columns are subjected to eccentric concentrated compressive loads (Fig. 2) as well as moments.
- The difference between a column and a beam is that in a column, the magnitude of \( P \) is much much greater than that of a beam.
The Secant Formula

- Denoting by $e$ the eccentricity of the load, that is, the distance between the line of action of $P$ and the axis of the column, as shown in Fig. 3a, the given eccentric load can be replaced by a centric force $P$ and a couple $M_A = Pe$ (Fig. 3b).
- It is clear that no matter how small $P$ and $e$, the couple $M_A$ will cause bending in the column, as shown in Fig. 3c.
The Secant Formula

- As the eccentric load is increased, both the couple $M_A$ and the axial force $P$ increase, and both cause the column to bend further.
- Viewed in this way, the problem of buckling is not a question of determining how long the column can remain straight and stable under an increasing load, but rather how much it can be permitted to bend under the increasing load, if the allowable stress is not to be exceeded and if the deflection $y_{\text{max}}$ is not to become excessive.
- The deflection equation (elastic curve) for this column can be written and solved in a manner similar to column subjected to centric loading $P$. 
The Secant Formula

Derivation of the formula:
- Drawing the free-body diagram of a portion AQ of the column of Figure 3 and choosing the coordinate axes as shown in Fig. 4, the bending moment at Q is given by

\[ M(x) = -Py - M_A = -Py - Pe \] (1)
Buckling: Eccentric Loading

- The Secant Formula
  - Derivation of the formula (cont’d):
    - Recalling that the relationship between the curvature and the moment along the column is given by
      \[
      \frac{d^2 y}{dx^2} = M(x) \quad (2)
      \]
    - Therefore, combining Eqs. 1 and 2, yields
      \[
      \frac{d^2 y}{dx^2} = -\frac{P}{EI} y - \frac{Pe}{EI} \quad (3)
      \]

- Moving the term containing \( y \) in Eq. 3 to the left and setting as done earlier, Eq. 3 can be written as
  \[
  \frac{d^2 y}{dx^2} + p^2 y = -p^2 e \quad (5)
  \]
Buckling: Eccentric Loading

- The Secant Formula
  - Derivation of the formula (cont’d):
    - The general solution of the differential equation (Eq. 5) is
      \[ y = A \sin px + B \cos px - e \]  \hspace{1cm} (6)
    - Using the boundary condition \( y = 0 \), at \( x = 0 \), Eq. 6 gives
      \[ B = e \]  \hspace{1cm} (7)

- Using the other boundary condition at the other end: \( y = 0 \), at \( x = L \), Eq. 6 gives
  \[ A \sin pL = e(1 - \cos pL) \]  \hspace{1cm} (8)

  Recalling that
  \[ \sin pL = 2 \sin \frac{pL}{2} \cos \frac{pL}{2} \]  \hspace{1cm} (9)
  and
  \[ 1 - \cos pL = 2 \sin^2 \frac{pL}{2} \]  \hspace{1cm} (10)
The Secant Formula

Derivation of the formula (cont’d):

- Substituting Eqs. 9 and 10 into Eq. 8, we obtain

\[ A = e \tan \frac{pL}{2} \]  \hspace{1cm} (11)

- And substituting for A and B into Eq. 6, the elastic curve can be obtained as

\[ y = e \left[ \tan \frac{pL}{2} \sin px + \cos px - 1 \right] \]  \hspace{1cm} (12)

The maximum deflection is obtained by setting \( x = L/2 \) in Eq. 12. The result is

\[ y_{\text{max}} = e \left[ \sec \frac{p}{\sqrt{EI/2}} - 1 \right] \]  \hspace{1cm} (13)

\( y_{\text{max}} \) becomes infinite when the argument of the secant function in Eq. 13 equals \( \pi/2 \). While the deflection does not become infinite, it nevertheless becomes unacceptably large, and \( P \) would reach the critical value \( P_{\text{cr}} \).
Buckling: Eccentric Loading

- **The Secant Formula**
  - **Derivation of the formula (cont’d):**
    - Based on that, when the argument of the secant function equals \( \pi/2 \), that is
      \[
      \sqrt{\frac{P}{EI}} \frac{L}{2} = \frac{\pi}{2}
      \]
      (14)
    - Therefore, the critical load is obtained as
      \[
      P_{cr} = \frac{\pi^2EI}{L^2}
      \]
      (15)
    - which is the same for column under centric loading.

- **Derivation of the formula (cont’d):**
  - Solving Eq. 15 for \( EI \) and substituting into Eq. 13, the maximum deflection can be expressed in an alternative form as
    \[
    y_{max} = e^{\sec \frac{\pi}{2} \left( \frac{P}{P_{cr}} \right)^{-1}}
    \]
    (16)
  - The maximum stress occurs at midspan of the column (at \( x = L/2 \)), and can be computed from
    \[
    \sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I}
    \]
    (17)
The Secant Formula

- Derivation of the formula (cont’d):
  - The moment at the midspan of the column is maximum ($M_{\text{max}}$) and is given by
    $$M_{\text{max}} = P y_{\text{max}} + M_A = P(y_{\text{max}} + e)$$  \hspace{1cm} (18)
  - Substituting this value of $M_{\text{max}}$ and the expression for $y_{\text{max}}$ of Eq. 13 into Eq. 17 and noting that $I = Ar^2$, the result is
    $$\sigma_{\text{max}} = \frac{P}{A} \left[ 1 + \frac{er}{r^2} \sec \left( \sqrt{\frac{P L}{EI}} \right) \right]$$  \hspace{1cm} (19)

- An alternative form of Eq. 19 is obtained by substituting for $y_{\text{max}}$ from Eq. 16 into Eq. 18. Thus
  $$\sigma_{\text{max}} = \frac{P}{A} \left[ 1 + \frac{ec}{r^2} \sec \left( \frac{P L}{P_{cr}} \right) \right]$$  \hspace{1cm} (20)
  - Where $P_{cr}$ is Euler buckling load for centric loading as given by Eq. 15.
The Secant Formula

- Derivation of the formula (cont’d):
  
  Notes on Equations 19 and 20:
  1. Since $\sigma_{\text{max}}$ does not vary linearly with $P$, the principle of superposition does not apply to the determination of the stress due to the simultaneous application of several loads.
  2. The resultant load must first be computed and then Eqs. 19 or 20 may be used to determine the corresponding stress.
  3. Consequently, any given factor of safety should be applied to the load and not to the stress.

If we substitute for $I = Ar^2$ in Eq. 19, and solve for $P/A$ in front of the bracket, the secant formula is obtained:

$$\frac{P}{A} = \frac{\sigma_{\text{max}}}{1 + \frac{ec}{r^2} \sec \left( \frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L'}{r} \right)}$$  \hspace{1cm} (21)

Where $L'$ is used to make the formula applicable to various end conditions.
Buckling: Eccentric Loading

The Secant Formula

The secant formula for a column subjected to eccentric compressive load \( P \) is given by

\[
\frac{P}{A} = \sigma_{\text{max}} \frac{1 + \frac{e c}{r^2} \sec \left( \frac{1}{2} \sqrt{\frac{P}{E A}} \right)}{1 + \frac{e c}{r^2} \sec \left( \frac{1}{2} \sqrt{\frac{P}{E A}} \right)}
\]

\( e = \) eccentricity \hspace{1cm} \( A = \) area of cross section

\( E = \) modulus of elasticity \hspace{1cm} \( r = \) min radius of gyration

\( c = \) distance from N.A. to extreme fiber.

\( L' = \) effective length for column

An alternative form for the secant formula is given by

\[
P = \sigma_{\text{max}} A \frac{1 + \frac{e c}{r^2} \sec \left( \frac{1}{2} \sqrt{\frac{P}{E A}} \right)}{1 + \frac{e c}{r^2} \sec \left( \frac{1}{2} \sqrt{\frac{P}{E A}} \right)}
\]

\( e = \) eccentricity \hspace{1cm} \( A = \) area of cross section

\( E = \) modulus of elasticity \hspace{1cm} \( r = \) min radius of gyration

\( c = \) distance from N.A. to extreme fiber.

\( L' = \) effective length for column
The Secant Formula

- General Notes on the Secant Formula:
  - The formula of Eq. 21 is referred to as the secant formula; it defines the force per unit area, $P/A$, which causes a specified maximum stress $\sigma_{\text{max}}$ in a column of given effective slenderness ratio $\lambda^2/r$, for a given value of the ratio $ec/r^2$, where $e$ is the eccentricity of the applied load.
  - Since $P$ or $P/A$ appears in both sides of Eqs. 21 or 22, it necessary to solve these equations by an iterative procedure (trial and error) to obtain the value of $P$ or $P/A$ corresponding to a given column and loading condition.
  - The curves shown in Figs. 5 and 6 are constructed for steel column using Eq. 21.
  - These curves make it possible to determine the load per unit area $P/A$, which causes the column to yield for given values of the ratio $\lambda^2/r$ and $ec/r^2$. 
Buckling: Eccentric Loading

**Figure 5.**
Load per unit area, $P/A$, causing yield in column (US Customary units)
(Beer & Johnston 1992)

**Figure 6.**
Load per unit area, $P/A$, causing yield in column (SI)
(Beer & Johnston 1992)
Buckling: Eccentric Loading

The Secant Formula

It should be noted that for small values of $L'/l r$, the secant is almost equal to unity in Eqs. 21 and 22, and $P/A$ (or $P$) may be assumed equal to

$$\frac{P}{A} = \frac{\sigma_{\text{max}}}{1 + \frac{ec}{r^2}}$$

or

$$P = \frac{\sigma_{\text{max}} A}{1 + \frac{ec}{r^2}}$$

(23)

Notes on Figures 5 and 6:

- For large value of $L'/l r$, the curves corresponding to the various values of the ratio $ec/r^2$ get very close to Euler's curve, and thus that the effect of the eccentricity of the loading on the value of $P/A$ becomes negligible.
- The secant formula is mainly useful for intermediate values of $L'/l r$. 
Buckling: Eccentric Loading

Example 1

The axial load $P$ is applied at a point located on the $x$ axis at a distance $e$ from the geometric axis of the W 250 $\times$ 58 rolled-steel column $AB$ (Fig. 7). When $P = 350$ kN, it is observed that the horizontal deflection of the top of the column is 5 mm. Using $E = 200$ GPa, determine (a) the eccentricity $e$ of the load, (b) the maximum stress in the column.

Example 1 (cont’d)

Figure 7
Buckling: Eccentric Loading

Example 1 (cont’d)

- For W 250 × 58 (see Handout or Fig. 8):
  \[ I_x = 87.0 \times 10^{-6} \text{ m}^2 \quad A = 7.42 \times 10^{-3} \text{ m}^2 \]
  \[ I_y = 18.73 \times 10^{-6} \text{ m}^2 \quad r_y = 0.1085 \text{ m} \]
  \[ S_y = 184.5 \times 10^{-6} \text{ m}^3 \quad r_y = 0.0502 \text{ m} \]
- One-end fixed, one-end free column,
  \[ L' = 2(3.2) = 6.4 \text{ m} \]
- Therefore
  \[ P_{cr} = \frac{\pi^2 EI}{L'^2} = \frac{\pi}{(6.4)^2} \left( 18.73 \times 10^{-6} \right) = 902.6 \text{ kN} \]
Example 1 (cont’d)

(a) From Eq. 16,

\[ y_{\text{max}} = e \left( \sec \frac{\pi}{2} \frac{P}{P_{\text{cr}}} - 1 \right) \]

\[ 0.005 = e \left( \sec \frac{\pi}{2} \frac{350}{902.6} - 1 \right) = \sec(0.9782) - 1 \]

\[ 0.005 = e \left( \frac{1}{\cos(0.9782)} - 1 \right) \Rightarrow e = 0.00633 \text{ m} = 6.33 \text{ mm} \]

(b) From Eq. 20,

\[ \sigma_{\text{max}} = \frac{P}{A} \left[ 1 + \frac{ec}{r^2} \sec \frac{\pi}{2} \frac{P}{P_{\text{cr}}} \right] = \]

\[ = \frac{350 \times 10^3}{7.42 \times 10^{-3}} \left[ 1 + \frac{(6.33) \left( \frac{203}{2} \times 10^{-3} \right)}{(0.0503)^3} \sec \frac{\pi}{2} \frac{350}{902.6} \right] \]

\[ = 68,615,044 \text{ N/m}^2 = 68.62 \text{ MPa} \]
Example 1 (cont’d)

- An alternate solution for Part (b):

\[
\begin{align*}
\gamma_m &= 5 \text{ mm} \\
e &= 6.33 \text{ mm} \\
P &= 350 \times 10^3 \text{ N} \\
M &= P(y_m + e) = 350[0.005 + 0.00633] = 3.966 \text{ kN - m} \\
\sigma_m &= \frac{P}{A} + \frac{M}{S_y} = \frac{350}{7.42 \times 10^{-3}} + \frac{3.966}{184.5 \times 10^{-6}} = 68.67 \text{ MPa}
\end{align*}
\]

Example 2

An axial load \( P \) is applied at a point located on the x-axis at a distance \( e = 0.60 \text{ in.} \) from the geometric axis of the W8 \( \times 28 \) rolled-steel column \( AC \) (Fig. 9). Knowing that the column is free at its top \( B \) and fixed at its base \( A \) and that \( \sigma_y = 36 \text{ ksi} \) and \( E = 29 \times 10^6 \text{ psi} \), determine the allowable load \( P \) if a factor of safety of 2.5 with respect to yield is required.
Example 2 (cont’d)

For W 8 × 28 (see Handout or Fig. 10):

- \( I_y = 21.7 \text{ in}^2 \quad A = 8.25 \text{ in}^2 \)
- \( c = \frac{6.535}{2} \text{ in} = 3.2675 \quad r_y = 1.62 \text{ in} \)
- One-end fixed, one-end free column, \( L’ = 2(6) = 12 \text{ ft} \)
- Therefore

\[
\frac{L'}{r_y} = \frac{12 \times 12}{1.62} = 88.99 \\
\frac{ec}{r^2} = \frac{0.6(3.2675)}{(1.62)^2} = 0.747
\]
Example 2 (cont’d)

- Since $\sigma_y = 36$ ksi, and $E = 29 \times 10^6$ ksi, Fig. 5 can be used:

  We read

  \[
  \frac{P}{A} \approx 15; \quad \text{therefore, } P = 15A = 15(8.25) = 123.75 \text{ kips}
  \]

- Thus

  \[
  P_{\text{allowable}} = \frac{P}{FS} = \frac{123.75}{2.5} = 49.5 \approx 50 \text{ kips}
  \]
Example 2 (cont’d)

- **General Iterative Procedure:**

  Suppose that we do not have the curves provided in Fig. 5, or we do have the curves but our problem consists of a column that has different material (e.g., \( \sigma_y = 50 \text{ ksi} \)), how can we evaluate the eccentric load \( P \) for Example 2?
Buckling: Eccentric Loading

Example 2 (cont’d)

A general trial and error (iterative) procedure can be used as follows:

- Using Eqs. 20 or 22, we assume an initial (guess) value for $P$ in the right-hand side of the equation; let it be 20 kips, hence

$$P = \frac{\sigma_{\text{max}} A}{1 + \frac{ec}{r^2 \sec \left( \frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L'}{r} \right)} \times 1 + 0.747 \left( \frac{1}{\cos \left( \frac{1}{2} \sqrt{\frac{20}{29 \times 10^3 \sigma_{\text{max}}}} \right)} \right) = 163.80 \text{ kips}$$

Example 2 (cont’d)

The revised value $P = 163.80 \text{ kips}$ can now be substituted in the right-hand side of the same equation to produce yet another revised value as follows:

$$P = \frac{\sigma_{\text{max}} A}{1 + \frac{ec}{r^2 \sec \left( \frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L'}{r} \right)} \times 1 + 0.747 \left( \frac{1}{\cos \left( \frac{1}{2} \sqrt{\frac{163.80}{29 \times 10^3 \sigma_{\text{max}}}} \right)} \right) = 103.01 \text{ kips}$$
Example 2 (cont’d)

A third iteration using a revised value for $P = 103.01$ kips, gives

$$P = \frac{\sigma_{\text{max}}A}{1 + \frac{ec}{r} \sec \left( \frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L'}{r} \right)} = \frac{36(8.25)}{1 + 0.747} \left( \frac{1}{\cos \left( \frac{1}{2} \sqrt{\frac{103.01}{29 \times 10^4}(8.25)} \right)} \right) \approx 132.79 \text{ kips}$$

Example 2 (cont’d)

– The iterative procedure is continued until the value of the eccentric load $P$ converges to the exact solution of 123.53 kips, as shown in the spreadsheet (Excel) result of Table 1.

– Therefore,

$$P_{\text{allowable}} = \frac{P}{FS} = \frac{123.53}{2.5} = 49.4 \approx 50 \text{ kips}$$
Buckling: Eccentric Loading

- Example 2 (cont’d)

Initial Value of $P$

$$P = \frac{\sigma_{\text{max}} A}{1 + \frac{e c}{r^2 \sec \left( \frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L'}{r} \right)} + \frac{P}{r}}$$

Table 1. Spreadsheet Result

<table>
<thead>
<tr>
<th>$P$ (kip)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.00</td>
<td>123.63</td>
</tr>
<tr>
<td>163.79</td>
<td>123.48</td>
</tr>
<tr>
<td>103.01</td>
<td>123.55</td>
</tr>
<tr>
<td>132.79</td>
<td>123.52</td>
</tr>
<tr>
<td>119.10</td>
<td>123.53</td>
</tr>
<tr>
<td>125.59</td>
<td>123.53</td>
</tr>
<tr>
<td>122.56</td>
<td>123.53</td>
</tr>
<tr>
<td>123.98</td>
<td>123.53</td>
</tr>
<tr>
<td>123.31</td>
<td>123.53</td>
</tr>
</tbody>
</table>