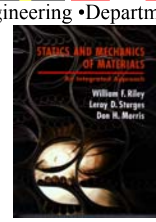
REVIEW FOR EXAM II



by

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SPRING 2002

ENES 220 – Mechanics of Materials

Department of Civil and Environmental Engineering

University of Maryland, College Park

Beams

- Introduction
  - – The most common type of structural member is a beam.
  - – In actual structures beams can be found in an infinite variety of
    - Sizes
    - Shapes, and
    - Orientations

Beams

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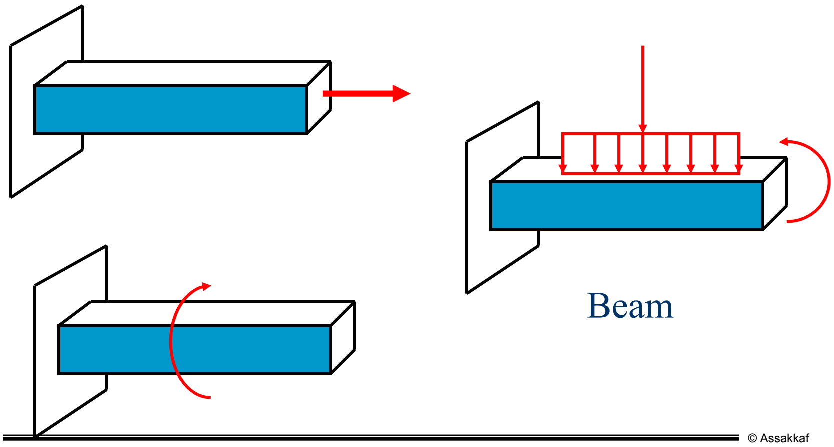
■ Introduction

Figure 1



Beams

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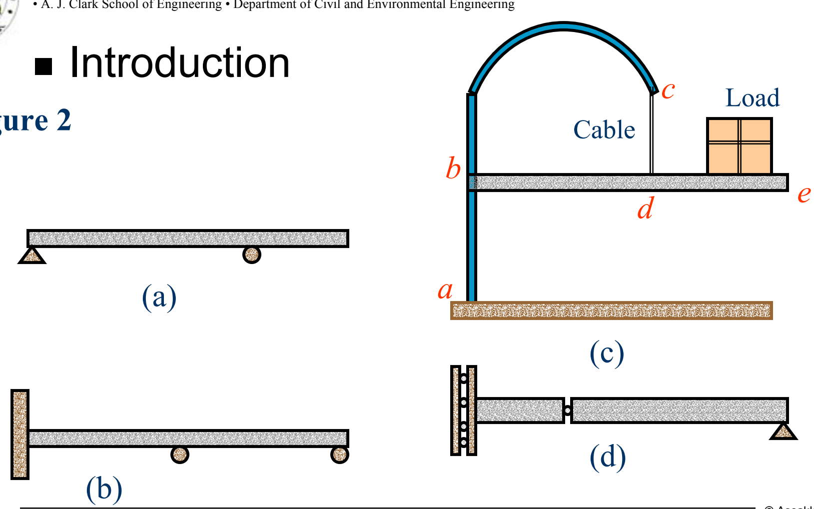
■ Introduction

Figure 2



Beams

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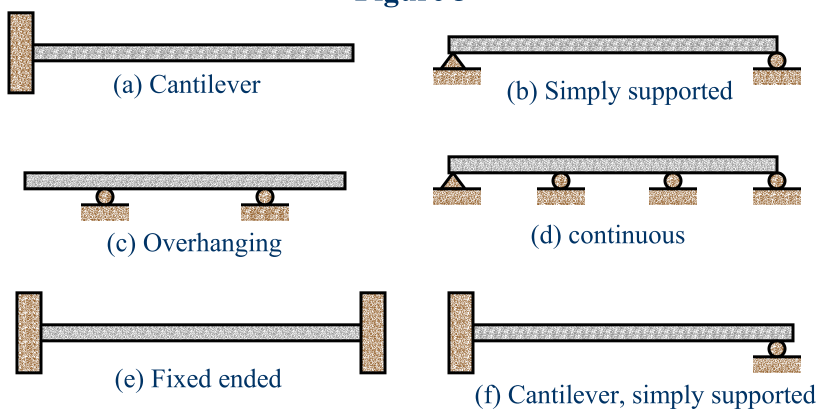
■ Introduction

Figure 3



Normal and Shearing Stress

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■ Stresses in beams

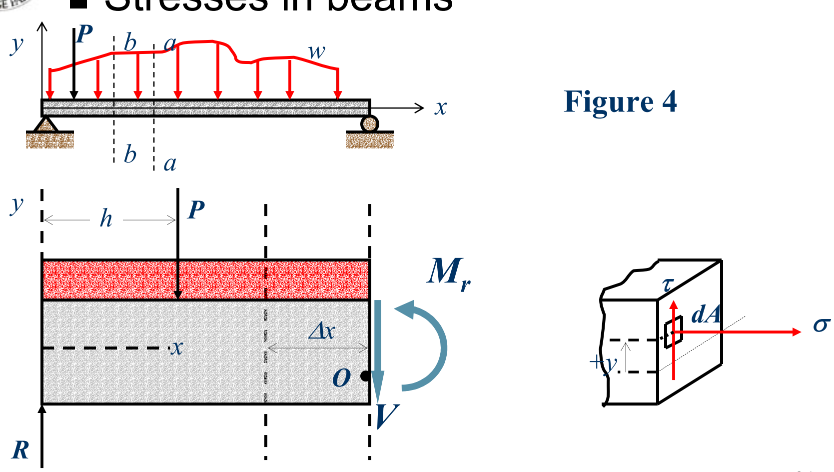
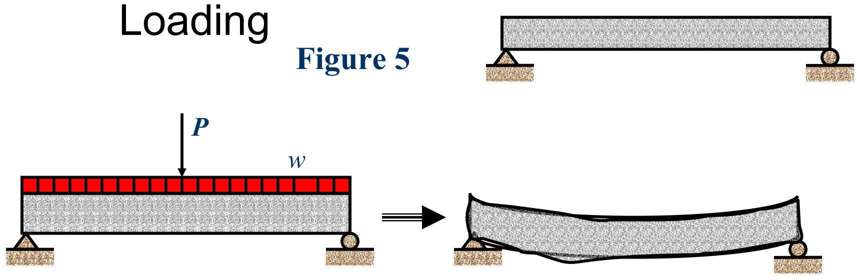


Figure 4

Flexural Strains

- Deformation of Beam due to Lateral Loading

Figure 5

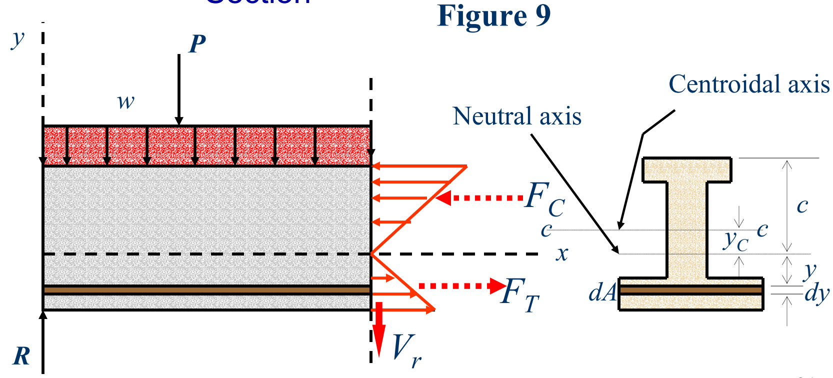


Flexural Stress

- Flexural Normal Stress

Distribution of Normal Stress in a Beam Cross Section

Figure 9



Flexural Stress

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- Flexural Normal Stress

For flexural loading and linearly elastic action, the neutral axis passes through the centroid of the cross section of the beam

Elastic Flexural Formula

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- The elastic flexural formula for normal stress is given by

$$\sigma_{\max} = \frac{M_r c}{I} \qquad (18)$$

and

$$\sigma_x = \frac{M_r y}{I} \qquad (19)$$

Elastic Flexural Formula

- An alternative form of the flexural formula for maximum normal stress is given by

$$\sigma_{\max} = \frac{M_r}{S} \qquad (20)$$

Where

$$S = \frac{I}{c}$$

Second Moments of Areas

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- Moment of Inertia
  - Consider an area *A* located in the *xy* plane as shown in the figure.

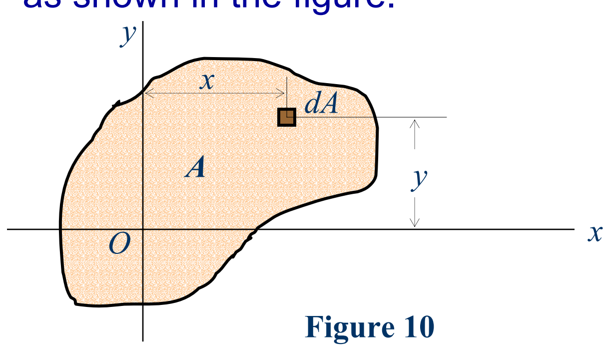
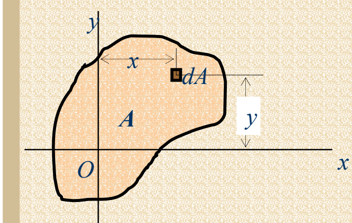


Figure 10

Second Moments of Areas

- Moment of Inertia



$$I_x = \int_A y^2 dA \qquad (22a)$$

$$I_y = \int_A x^2 dA \qquad (22b)$$

Where
I_x = moment of inertia with respect to x axis
I_y = moment of inertia with respect to y axis

Second Moments of Areas

- Radii of Gyration of an Area

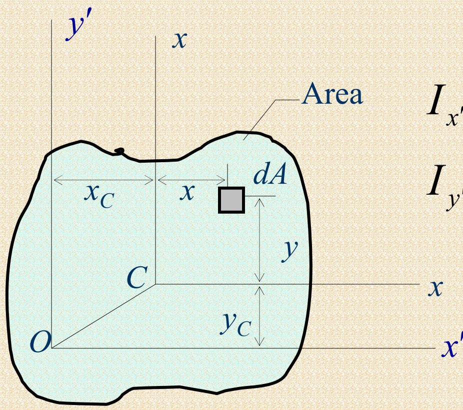
$$k_x = \sqrt{\frac{I_x}{A}} \qquad (26a)$$

$$k_y = \sqrt{\frac{I_y}{A}} \qquad (26b)$$

$$k_z = \sqrt{\frac{I_z}{A}} \qquad (26c)$$

Second Moments of Areas

- Parallel Axis Theorem



$$I_{x'} = I_{x_C} + y_c^2 A$$
$$I_{y'} = I_{y_C} + x_C^2 A \qquad (27)$$

Examples: Elastic Flexure Formula

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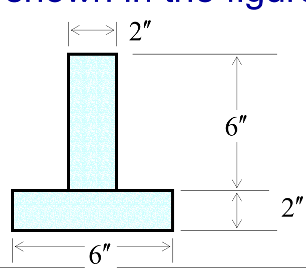
- Example 3

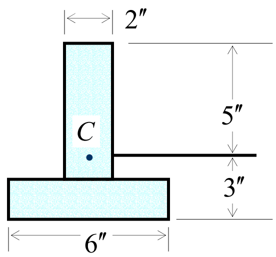
Determine the maximum flexural stress produced by a resisting moment M_r of +5000 ft·lb if the beam has the cross section shown in the figure.



Examples: Elastic Flexure Formula

- Example 3 (cont'd)

  First, we need to locate the neutral axis from the bottom edge:



$$y_C = \frac{(1)(2\times 6)+(2+3)(2\times 6)}{2\times 6+2\times 6} = \frac{72}{24} = 3''$$

$$y_{\text{ten}} = 3'' \qquad y_{\text{com}} = 6+2-3 = 5'' = y_{\max}$$

$$\text{Max. Stress} = \frac{M_r y_{\max}}{I_x}$$

Examples: Elastic Flexure Formula

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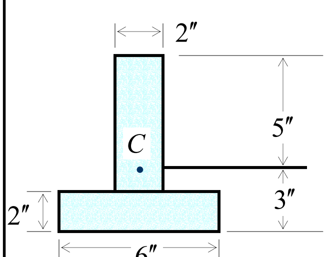
- Example 3 (cont'd)

  Find the moment of inertia I_x with respect to the *x* axis using parallel axis-theorem:



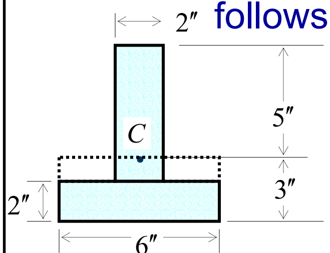
$$I_x = \frac{6(2)^3}{12} + (6\times 2)(2)^2 + \frac{2(6)^3}{12} + (2\times 6)(3-1)^2$$

$$= 4+48+36+48 = 136\ \text{in}^4$$

$$\text{Max. Stress (com)} = \frac{(5\times 12)(5)}{136} = \underline{2.21\ \text{ksi}}$$

Examples: Elastic Flexure Formula

- Example 3 (cont'd)
  – An alternative way for finding the moment of inertia I_x with respect to the x axis is as follows:

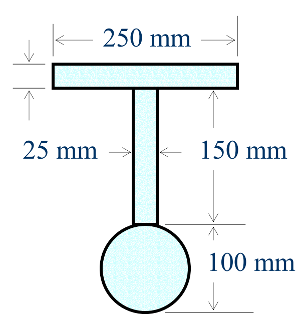


$$I_x = \frac{6(3)^3}{3} + \frac{2(5)^3}{3} - 2\left[\frac{2(1)^3}{3}\right] = 136$$

Examples: Elastic Flexure Formula

- Example 5

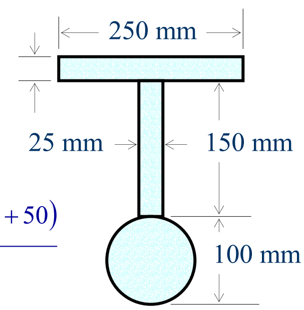
  Determine both the maximum flexural tensile and the maximum flexural compressive stresses produced by a resisting moment of 100 kN·m if the beam has the cross section shown in the figure.



Examples: Elastic Flexure Formula

- Example 5 (cont'd)

Locate the neutral axis from the upper edge:



$$y_C = \frac{250\times 25(12.5)+150\times 25(25+75)+\dfrac{\pi(100)^2}{4}(25+150+50)}{250\times 25+150\times 25+\dfrac{\pi(100)^2}{4}}$$

$$= \frac{2{,}220{,}270.87}{17{,}853.90}$$

$$= 124.36 \text{ mm}$$

Examples: Elastic Flexure Formula

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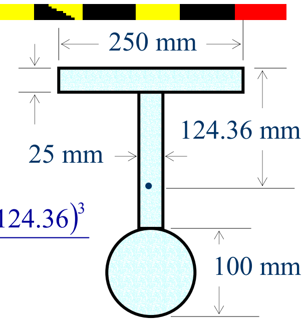
- Example 5 (cont'd)

Calculate the moment of inertia with respect to the *x* axis:



$$I_x = \frac{250(124.36)^3}{3} - \frac{(250-25)(124.36-25)^3}{3} + \frac{25(175-124.36)^3}{3} + \frac{\pi(100)^4}{64} + \frac{\pi(100)^2}{4}(225-124.36)^2$$

$$= 172.243\times 10^6 \text{ mm}^4 = 172.243\times 10^{-6} \text{ m}^4$$

$$\sigma_{\max}(\text{ten}) = \frac{M_r y}{I} = \frac{100\times 10^3(275-124.36)\times 10^{-3}}{172.243\times 10^{-6}} = \underline{87.5 \text{ MPa}}$$

$$\sigma_{\max}(\text{com}) = \frac{100\times 10^3(124.36)\times 10^{-3}}{172.243\times 10^{-6}} = \underline{72.2 \text{ MPa}}$$

Shear Forces and Bending Moments in Beams

- Variation of Shear and Moment Forces
  - In general, the internal shear V and bending moment M variations will be discontinuous, or their slope will be discontinuous at points where a distributed load changes or where concentrated forces or couples are applied.

Shear Forces and Bending Moments in Beams

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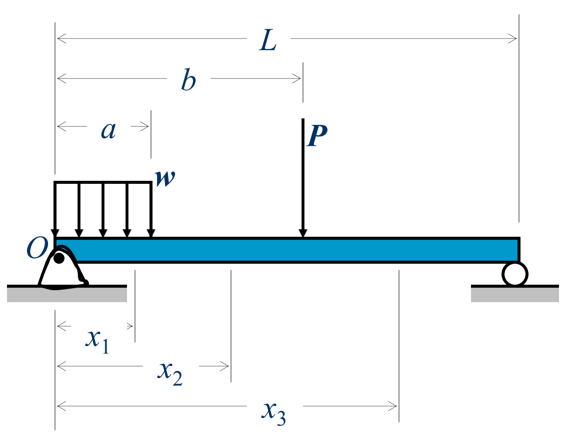
- Variation of Shear and Moment Forces

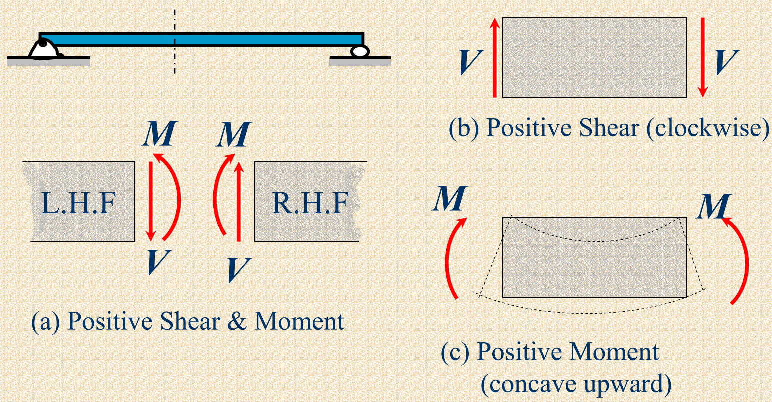
Figure 14



Shear Forces and Bending Moments in Beams

■ Sign Convention

Figure 15



(a) Positive Shear & Moment

(b) Positive Shear (clockwise)

(c) Positive Moment (concave upward)

Shear Forces and Bending Moments in Beams

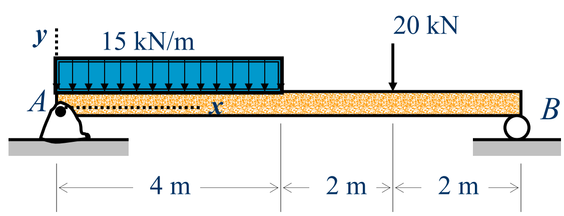
■ Sign Convention

– Perhaps an easy way to remember this sign convention is to isolate a small beam segment and note that *positive shear* tends to rotate the segment clockwise (Fig. 15b), and a *positive moment* tends to bend the segment concave upward (Fig. 15c)

Shear Forces and Bending Moments in Beams

- Example 6

  A beam is loaded and supported as shown in the figure. Using the coordinate axes shown, write equations for shear *V* and bending moment *M* for any section of the beam in the interval 0 < *x* < 4 m.



Shear Forces and Bending Moments in Beams

- Example 6 (cont'd)
  - A free-body diagram for the beam is shown Fig. 16. The reactions shown on the diagram are determined from equilibrium equations as follows:

$$+\circlearrowleft \sum M_B = 0; R_A(8) - (15\times 4)(6) - 20(2) = 0$$

$$\therefore R_A = 50 \text{ kN}$$

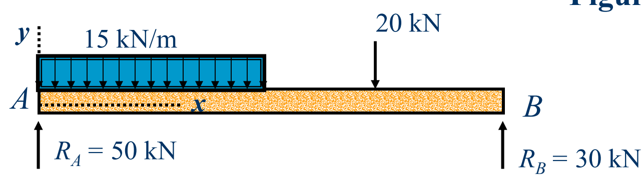
$$+\uparrow \sum F_y - 0; R_B + 50 - 15(4) - 20 = 0$$

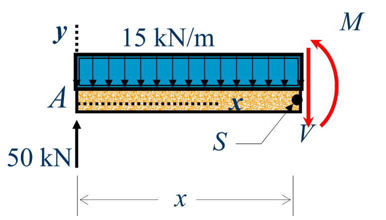
$$\therefore R_B = 30 \text{ kN}$$

Shear Forces and Bending Moments in Beams

- Example 6 (cont'd)

Figure 16





$$+\uparrow \sum F_y = 0; -V + 50 - 15x = 0$$

$$\therefore \boxed{V = 50 - 15x \qquad \text{for } 0 < x < 4}$$

$$+\circlearrowleft \sum M_S = 0; -M + 50x - 15(x)\frac{x}{2} = 0$$

$$\therefore \boxed{M = 50x - 7.5x^2 \qquad \text{for } 0 < x < 4}$$

Shear Forces and Bending Moments in Beams

- Load, Shear Force, and Bending Moment Relationships
  - In cases where a beam is subjected to several concentrated forces, couples, and distributed loads, the equilibrium approach discussed previously can be tedious because it would then require several cuts and several free-body diagrams.
  - In this section, a simpler method for constructing shear and moment diagrams are discussed.

Shear Forces and Bending Moments in Beams

- Load and Shear Force Relationships

$$\frac{dV}{dx} = w(x) \qquad (33)$$

Slope of Shear Diagram = Distributed Load Intensity

Shear Forces and Bending Moments in Beams

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- Load and Shear Force Relationships

$$\Delta V_{2-1} = V_2 - V_1 = \int_{V_1}^{V_2} dV = \int_{x_1}^{x_2} w(x)\, dx \qquad (34)$$

Change in Shear = Area under Loading Curve between x_1 and x_2

Shear Forces and Bending Moments in Beams

- Shear Force and Bending Moment Relationships

$$\frac{dM}{dx} = V \qquad (40)$$

Slope of Moment Diagram = Shear

Shear Forces and Bending Moments in Beams

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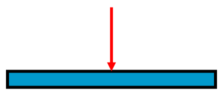
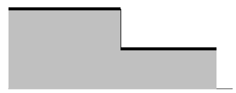
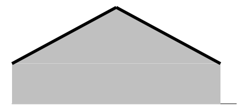
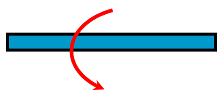
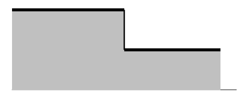
- Shear Force and Bending Moment Relationships

$$\Delta M_{2-1} = M_2 - M_1 = \int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} V\, dx \qquad (41)$$

Change in Moment = Area under Shear diagram between x_1 and x_2

Shear Forces and Bending Moments in Beams

■ Shear and Moment Diagrams

Loading	Shear Diagram, $\frac{dV}{dx} = w$	Moment Diagram, $\frac{dM}{dx} = V$
		
		

Shear Forces and Bending Moments in Beams

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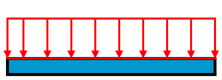
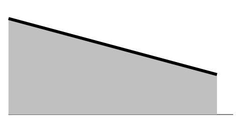
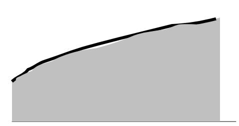
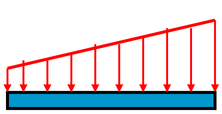
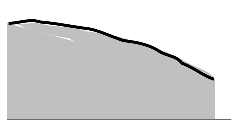
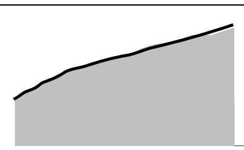
■ Shear and Moment Diagrams

Loading	Shear Diagram, $\frac{dV}{dx} = w$	Moment Diagram, $\frac{dM}{dx} = V$
		
		

Shear and Moment Diagrams

- Example 9

  Draw the shear and bending moment diagrams for the beam shown in Figure 21a.

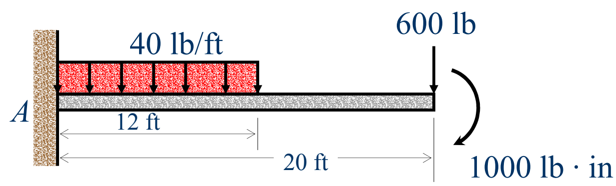


Figure 21a

Shear and Moment Diagrams

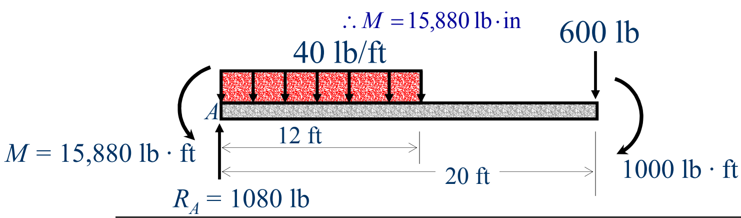
- Example 9 (cont'd)
  - Support Reactions
    - The reactions at the fixed support can be calculated as follows:

$$+\uparrow \sum F_y = 0;\ R_A - 40(12) - 600 = 0 \rightarrow R_A = 1080 \text{ lb}$$

$$+ \sum M_A = 0;\ -M + 40(12)(6) + 600(20) + 1000 = 0$$

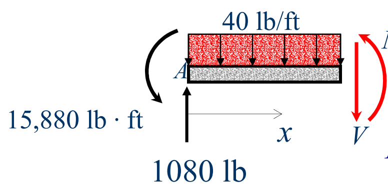
$$\therefore M = 15{,}880 \text{ lb} \cdot \text{in}$$

Figure 21b



Shear and Moment Diagrams

- Example 9 (cont'd)
  - Shear Diagram
    - Using the established sign convention, the shear at the ends of the beam is plotted first. For example, when *x* = 0, *V* = 1080; and when *x* = 20, *V* = 600

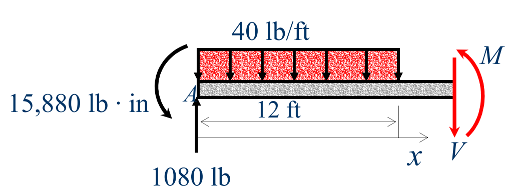


$$V(x) = 1080 - 40x \quad \text{for } 0 < x < 12$$

$$M = 1080x - 40\frac{x^2}{2} - 15{,}880 \quad \text{for } 0 < x < 12$$

Shear and Moment Diagrams

- Example 9 (cont'd)

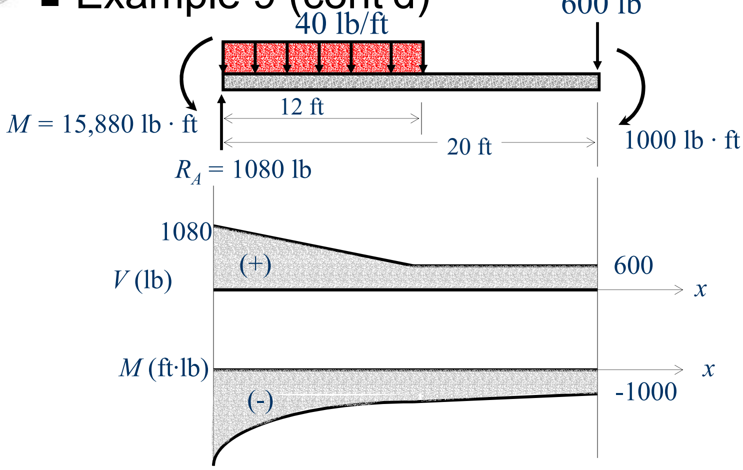


$$V(x) = 600 \qquad \text{for } 12 < x < 20$$

$$M = -15{,}880 - 40(12)(x - 6) + 1080x \qquad \text{for } 12 < x < 20$$

Shear and Moment Diagrams

- Example 9 (cont'd)



Shear and Moment Diagrams

- Example 10

Draw complete shear and bending moment diagrams for the beam shown in Fig. 22

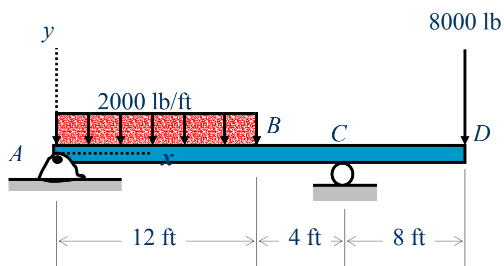


Figure 22a

Shear and Moment Diagrams

- Example 10 (cont'd)
  - The support reactions were computed from equilibrium as shown in Fig. 22.b.

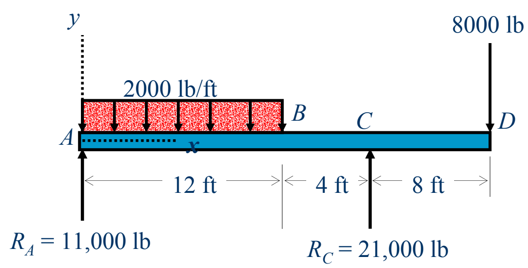
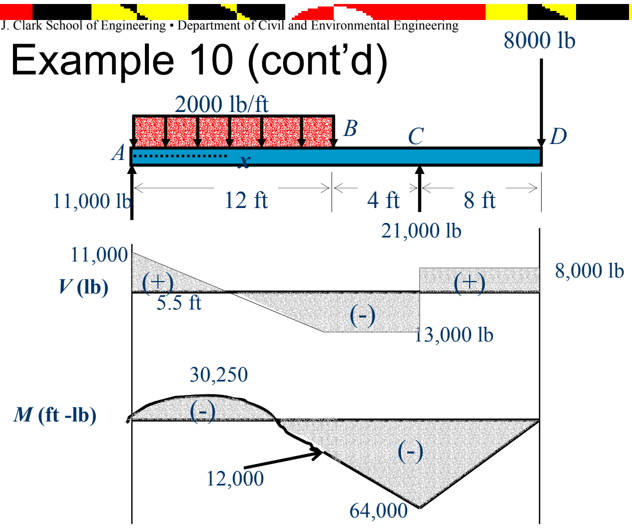


Figure 22a

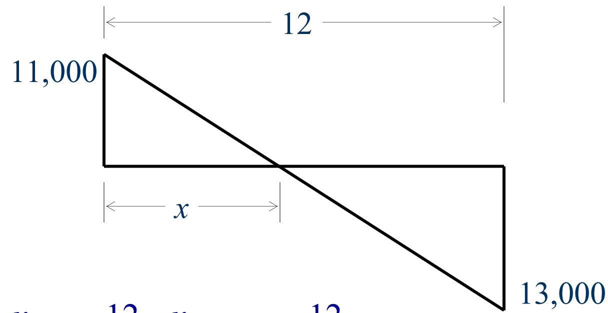
Shear and Moment Diagrams

- Example 10 (cont'd)



Shear and Moment Diagrams

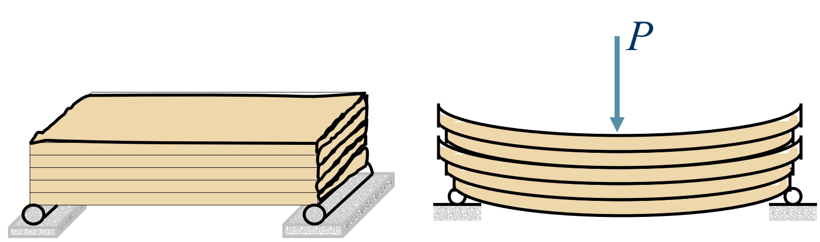
- Example 10 (cont'd)



$$\frac{x}{11{,}000} = \frac{12 - x}{13{,}000} \Rightarrow x = \frac{12}{2.18} = 5.5$$

Shearing Stress in Beams

- Shearing Stress due to Bending

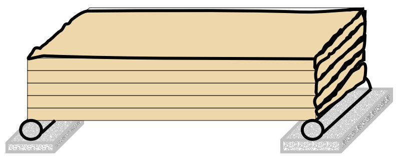


(a) Unloaded Stack of Slabs **(b) Unglued Slabs loaded**

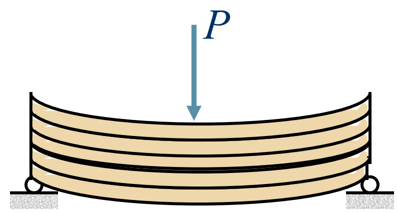
Figure 22

Shearing Stress in Beams

- Shearing Stress due to Bending



(c) Glued Slabs Unloaded



(d) Glued Slabs loaded

Figure 22 (cont'd)

Shearing Stress in Beams

- Shearing Stress Formula

At each point in the beam, the horizontal and vertical shearing stresses are given by

$$\tau = \frac{VQ}{It} \qquad (52)$$

Where
V = shear force at a particular section of the beam
Q = first moment of area of the portion of the cross-sectional area between the transverse line where the stress is to be computed.
I = moment of inertia of the cross section about neutral axis
t = average thickness at a particular location within the cross section

Shearing Stress in Beams

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- Example 11

  Determine the first moment of area Q for the areas indicated by the shaded areas *a* and *b* of Fig. 25.

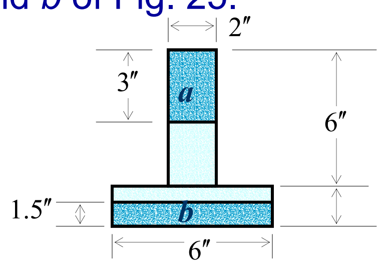

Figure 25

Shearing Stress in Beams

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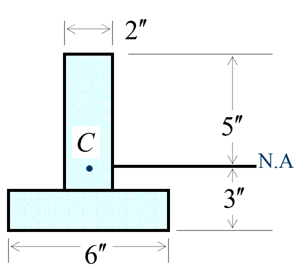
- Example 11 (cont'd)

  First, we need to locate the neutral axis from the bottom edge:

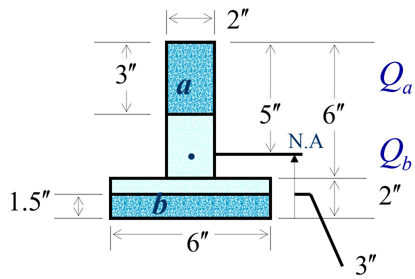


$$y_C = \frac{(1)(2\times 6)+(2+3)(2\times 6)}{2\times 6+2\times 6} = \frac{72}{24} = 3'' \text{ from base}$$

Shearing Stress in Beams

- Example 11 (cont'd)

The first moments of area Q_a and Q_b are found as follows:



$$Q_a = (5 - 1.5)[3 \times 2] = 21\ \text{in}^3$$

$$Q_b = \left(3 - \frac{1.5}{2}\right)[1.5 \times 6] = 20.25\ \text{in}^3$$

Shearing Stress in Beams

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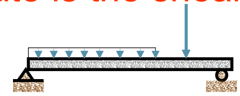
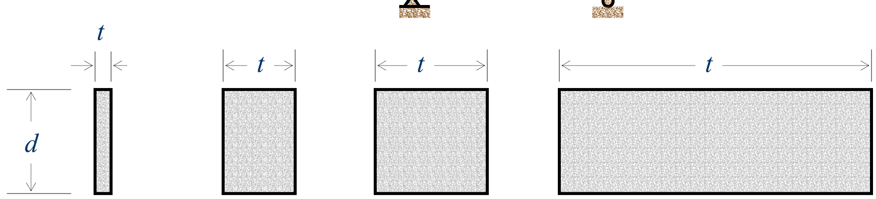
- Shearing Stress Formula

How accurate is the shearing stress formula?





Great
t is small

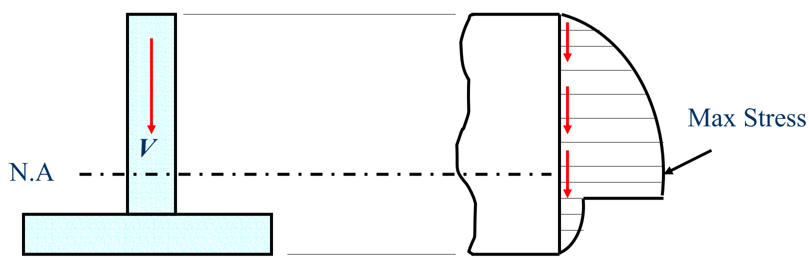
3% error
$d = 2t$

12% error
$d = t$

100% error, worst case
$4d = t$

Shearing Stress in Beams

- Variation of Vertical Shearing Stress in the Cross Section



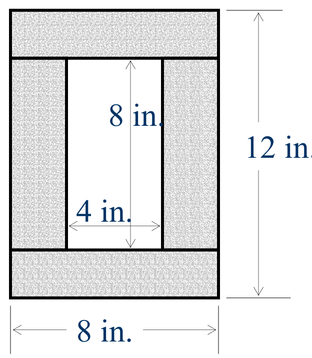
Shearing Stress in Beams

- Example 13

  The transverse shear *V* at a certain section of a timber beam is 600 lb. If the beam has the cross section shown in the figure, determine (a) the vertical shearing stress 3 in. below the top of the beam, and (b) the maximum vertical stress on the cross section.

Shearing Stress in Beams

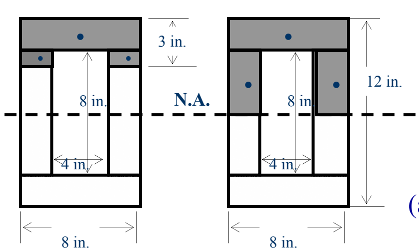
- Example 13 (cont'd)



Shearing Stress in Beams

- Example 13 (cont'd)

From symmetry, the neutral axis is located 6 in. from either the top or bottom edge.



$$I = \frac{8(12)^3}{12} - \frac{4(8)^3}{12} = 981.3\ \text{in}^4$$

$$Q_{3''} = 8(2)(5) + 2[1(2)(3.5)] = 94.0\ \text{in}^3$$

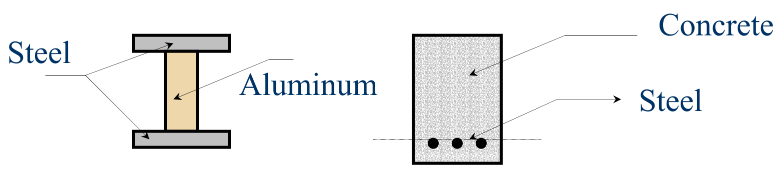
$$Q_{NA} = 8(2)(5) + 2[2(2)(4)] = 112.0\ \text{in}^3$$

$$\text{(a)}\ \tau_{Q_{3''}} = \frac{VQ_{3''}}{It} = \frac{6000(94)}{981.3(4)} = \underline{143.7\ \text{psi}}$$

$$\text{(b)}\ \tau_{\max} = \frac{VQ_{\max}}{It} = \frac{6000(112)}{981.3(4)} = \underline{171.2\ \text{psi}}$$

Composite Beams

- Bending of Composite Beams
  - These are called ***composite beams***.
  - They offer the opportunity of using each of the materials employed in their construction advantage.



Composite Beams

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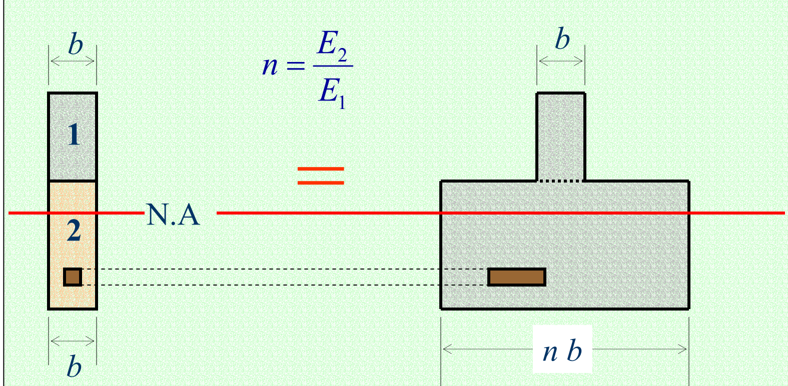
- Transformed Section

Figure 29



Composite Beams

- Transformed Section

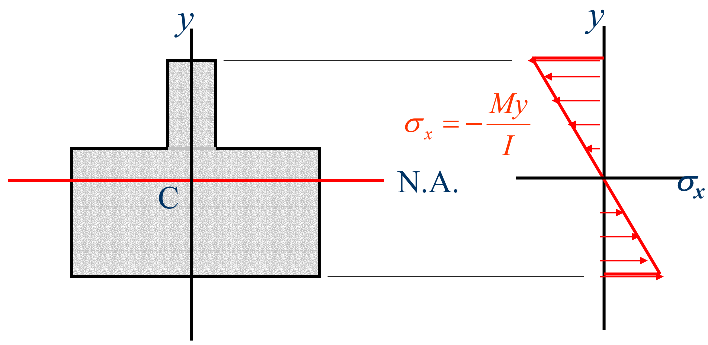


Figure 30. Distribution of Fictitious Normal Stress on Cross Section

Composite Beams

- Stresses on Transformed Section
  1. To obtain the stress σ_1 at a point located in the upper portion of the cross section of the original composite beam, the stress is simply computed from *My/I*.
  2. To obtain the stress σ_2 at a point located in the upper portion of the cross section of the original composite beam, stress σ_x computed from *My/I* is multiplied by *n*.

Composite Beams

- Example 17

A steel bar and aluminum bar are bonded together to form the composite beam shown. The modulus of elasticity for aluminum is 70 GPa and for streel is 200 GPa. Knowing that the beam is bent about a horizontal axis by a moment *M* = 1500 N-m, determine the maximum stress in (a) the aluminum and (b) the steel.

Composite Beams

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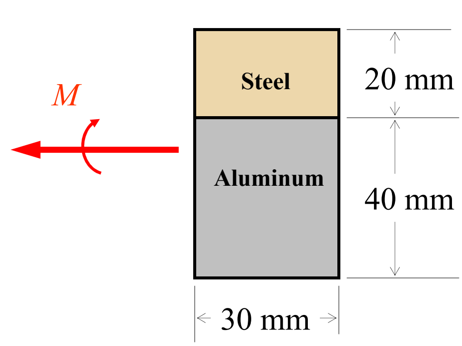
- Example 17 (cont'd)



Composite Beams

- Example 17 (cont'd)

  First, because we have different materials, we need to transform the section into a section that represents a section that is made of homogeneous material, either steel or aluminum.

  We have

$$n = \frac{E_s}{E_a} = \frac{200}{70} = 2.857$$

Composite Beams

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- Example 17 (cont'd)

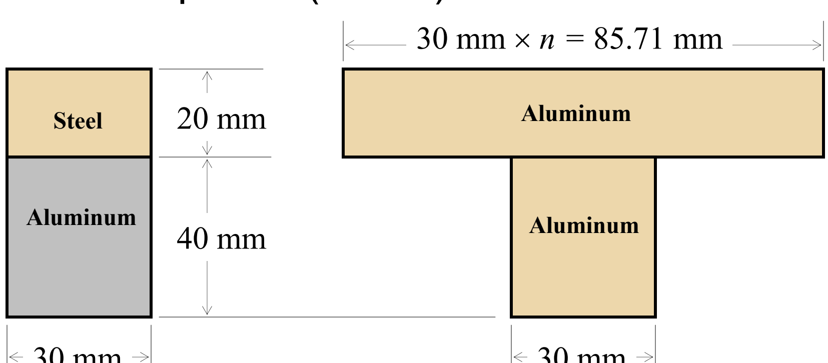

Figure 31a **Figure 31b**

Composite Beams

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- Example 17 (cont'd)

  Consider the transformed section of Fig. 31b, therefore

$$y_C = \frac{10(85.71\times 20)+40(30\times 40)}{(85.71\times 20)+(30\times 40)} = 22.353 \text{ mm from top}$$

$$I_{NA} = \frac{85.71(22.353)^3}{3} - \frac{(85.71-30)(22.353-20)^3}{3}$$

$$+\frac{30(40+20-22.353)^3}{3} = 852.42\times 10^3 \text{ mm}^4 = 852.42\times 10^{-9} \text{ m}^4$$

Composite Beams

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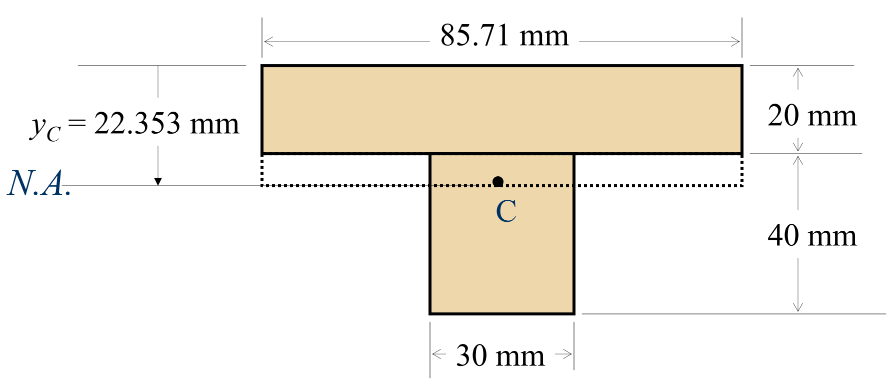
- Example 17 (cont'd)



Composite Beams

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- Example 17 (cont'd)

  a) Maximum normal stress in aluminum occurs at extreme lower fiber of section, that is at *y* = -(20+40-22.353) = -37.65 mm.

$$\sigma_{al} = -\frac{My}{I} = -\frac{1500\left(-37.65\times10^{-3}\right)}{852.42\times10^{-9}} = 66.253\times10^{6}\,\text{Pa}$$

$$\boxed{= +66.253\ \text{MPa (T)}}$$

Composite Beams

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- Example 17 (cont'd)

  b) Maximum normal stress in stelel occurs at extreme upper fiber of the cross section, that is. at *y* =+ 22.353 mm.

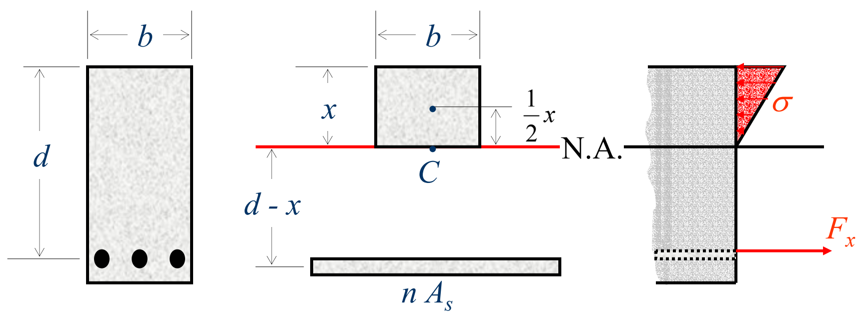
$$\sigma_{St} = -n\frac{My}{I} = -(2.867)\frac{1500\left(22.353\times10^{-3}\right)}{852.42\times10^{-9}} = -112.8\times10^{6}\,\text{Pa}$$

$$= 112.8\ \text{MPa (C)}$$

Composite Beams

- Reinforced Concrete Beam

Figure 33



(a) **(b)** **(c)**

Composite Beams

- Reinforced Concrete Beam
  - The ratio *n* is given by

$$n = \frac{\text{Modulus of Elasticity for Steel}}{\text{Modulus of Elasticity for Concrete}} = \frac{E_s}{E_c}$$

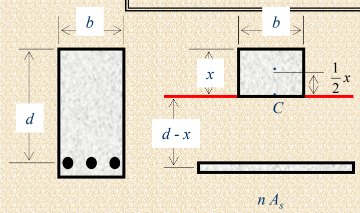
  - The position of the neutral axis is obtained by determining the distance *x* from the upper face of the beam (upper fiber) to the centroid *C* of the transformed section.

Composite Beams

- Reinforced Concrete Beam

  The neutral axis for a concrete beam is found by solving the quadratic equation:

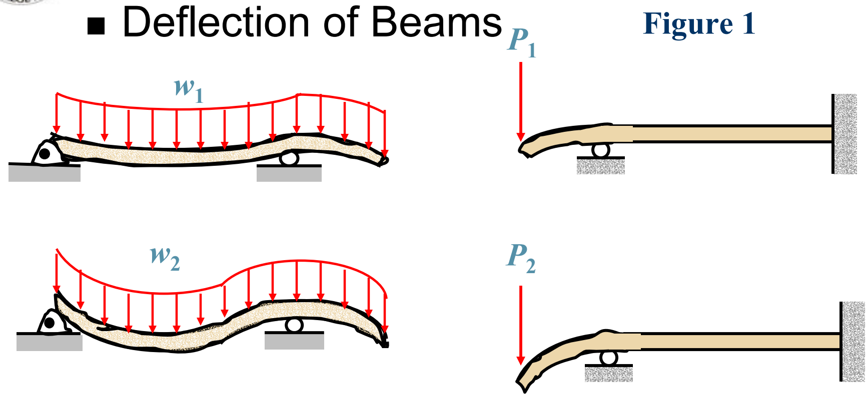
$$\boxed{\frac{1}{2}bx^2 + nA_s x - nA_s d = 0} \quad (62)$$



Beam Deformation

- Deflection of Beams

Figure 1



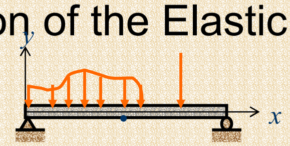
(a) $w_2 >> w_1$ (b) $P_2 >> P_1$

Beam Deformation

- Methods for Determining Beam Deflections
  - Three methods are commonly used to find beam deflections:
    1) The double integration method,
    2) The singularity function method, and
    3) The superposition method

Beam Deformation

- The Differential Equation of the Elastic Curve for a Beam

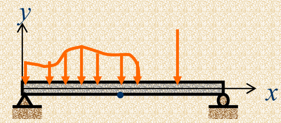


$$EI\frac{d^2y}{dx^2} = M(x) \qquad (8)$$

E = modulus of elsticity for the material
I = moment of inertia about the neutral axis of cross section
$M(x)$ = bending moment along the beam as a function of x

Beam Deformation

■ Sign Convention



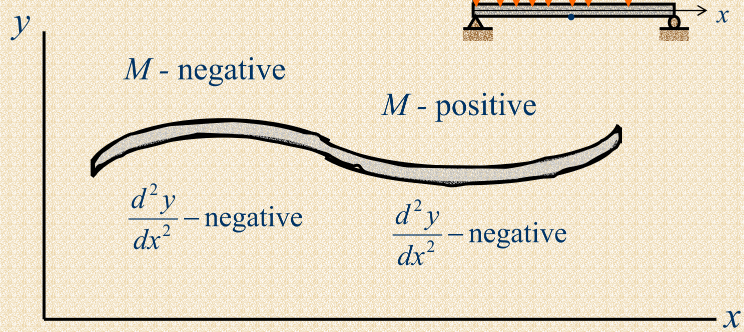
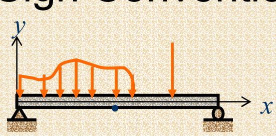


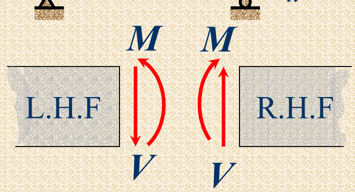
Figure 5. Elastic Curve

Beam Deformation

■ Sign Convention

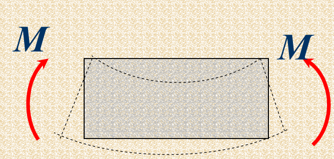
Figure 6





(a) Positive Shear & Moment

(b) Positive Shear (clockwise)



(c) Positive Moment
(concave upward)

Beam Deformation

- Relation of the Deflection y with Physical Quantities such as V and M

$$
\begin{aligned}
\text{deflection} &= y \\
\text{slope} &= \frac{dy}{dx} \\
\text{moment}\,(M) &= EI\frac{d^2y}{dx^2} \\
\text{shear}\,(V) &= \frac{dM}{dx} = EI\frac{d^3y}{dx^3}\ (\text{for } EI \text{ constant}) \\
\text{load}\,(w) &= \frac{dV}{dx} = EI\frac{d^4y}{dx^4}\ (\text{for } EI \text{ constant})
\end{aligned}
\qquad (9)
$$

Beam Deformation

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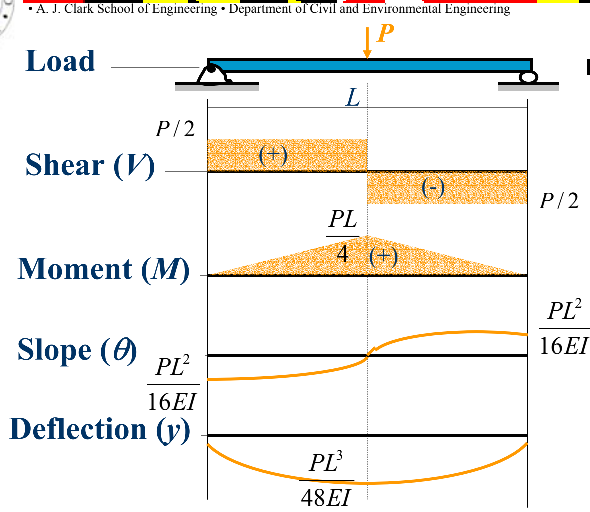

- Figure 7
  - Complete Series of Diagrams for Simply Supported beam

Deflection by Integration

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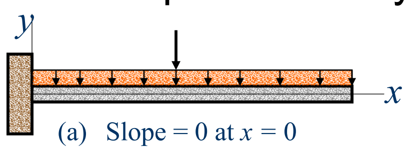
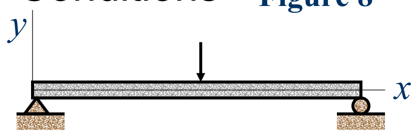
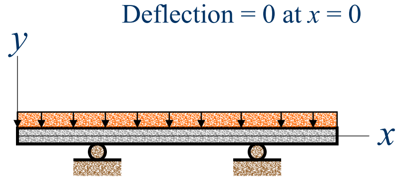
■ Example Boundary Conditions **Figure 8**



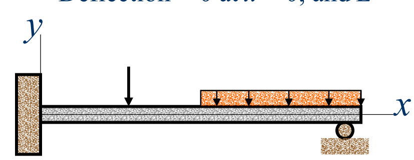
(a) Slope = 0 at x = 0
Deflection = 0 at x = 0



(b) Slope at $L/2$ = 0
Deflection = 0 at x = 0, and L



(c) Slope at rollers ?
Deflection at rollers = 0



(d) Slope = 0 at x =0
Deflection = 0 at x = 0 and x = L

Deflection by Integration

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- Example 2

  A beam is loaded and supported as shown in the figure.

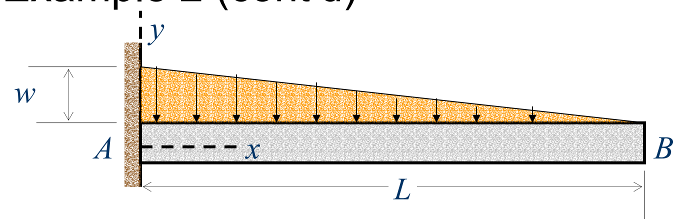
  a) Derive the equation for the elastic curve in terms of w, L, x, E, and I.

  b) Determine the slope at the right end of the beam.

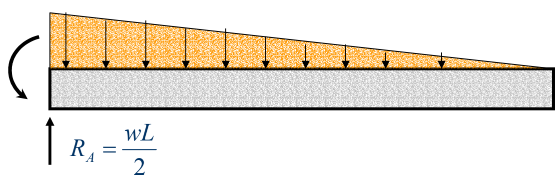
  c) Find the deflection at x = L.

Deflection by Integration

- Example 2 (cont'd)



FBD



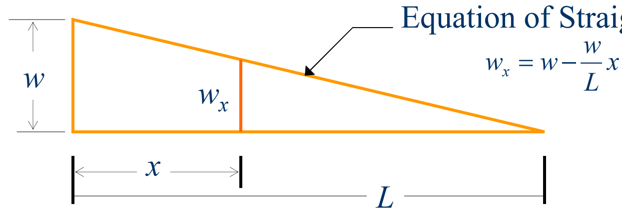
$$M_A = \frac{wL}{2}\left(\frac{2L}{3}\right) = \frac{wL^2}{6}$$

$$R_A = \frac{wL}{2}$$

Deflection by Integration

- Example 2 (cont'd)

Find an expression for a segment of the distributed load:



$$w_x = w - \frac{w}{L}x$$

$$\frac{w_x}{L-x} = \frac{w}{L} \quad \Rightarrow \quad \boxed{w_x = \frac{w(L-x)}{L} = w - \frac{w}{L}x} \qquad (13a)$$

Deflection by Integration

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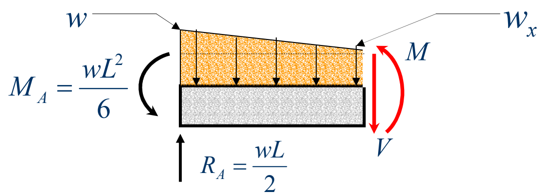
- Example 2 (cont'd)



$$+\circlearrowleft \sum M_s = 0; -M - \frac{wL^2}{6} + \frac{wL}{2}x - (w_x x)\frac{x}{2} - \frac{(w-w_x)x}{2}\frac{2x}{3} = 0$$

or

$$M(x) = -\frac{wL^2}{6} + \frac{wL}{2}x - \frac{w_x}{2}x^2 - \frac{(w-w_x)x^2}{3} \qquad (13b)$$

Deflection by Integration

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- Example 2 (cont'd)
  - The solution for parts (a), (b), and (c) can be completed by substituting for w_x into Eq. 13b, equating the expression for $M(x)$ to the term $EI(d^2y/dx^2)$, and integrating twice to get the elastic curve and expression for the slope.
  - Note that the boundary conditions are that both the slope and deflection are zero at x = 0. i.e.; $EIy'' = EI\frac{d^2y}{dx^2} = M(x)$

Singularity Functions

- Definition

> A singularity function is an expression for *x* written as $\langle x - x_0 \rangle^n$, where *n* is any integer (positive or negative) including zero, and x_0 is a constant equal to the value of *x* at the initial boundary of a specific interval along the beam.

Singularity Functions

- Properties of Singularity Functions
  - By definition, for $n \geq 0$,

$$\langle x - x_0 \rangle^n = \begin{cases} (x - x_0)^n & \text{when } x \geq x_0 \\ 0 & \text{when } x < x_0 \end{cases} \qquad (16)$$

  - Selected properties of singularity functions that are useful and required for beam-deflection problems are listed in the next slides for emphasis and ready reference.

Singularity Functions

- Selected Properties

$$\langle x - x_0 \rangle^n = \begin{cases} (x - x_0)^n & \text{when } n > 0 \text{ and } x \geq x_0 \\ 0 & \text{when } n > 0 \text{ and } x < x_0 \end{cases} \quad (17)$$

$$\langle x - x_0 \rangle^0 = \begin{cases} 1 & \text{when } n > 0 \text{ and } x \geq x_0 \\ 0 & \text{when } n > 0 \text{ and } x < x_0 \end{cases} \quad (18)$$

Singularity Functions

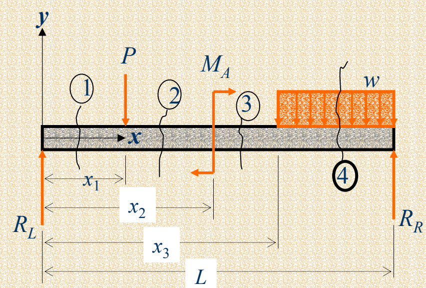
- Integration and Differentiation of Singularity Functions

$$\int \langle x - x_0 \rangle^n dx = \frac{1}{n+1} \langle x - x_0 \rangle^{n+1} + \text{C} \quad \text{when } n > 0 \quad (19)$$

$$\frac{d}{dx} \langle x - x_0 \rangle^n = n \langle x - x_0 \rangle^{n-1} \quad \text{when } n > 0 \quad (20)$$

Singularity Functions

- Typical Singularity Functions



$$M(x) = R_L x - P\langle x - x_1\rangle^1 + M_A\langle x - x_2\rangle^0 - \frac{w}{2}\langle x - x_3\rangle^2 \qquad \text{for } 0 < x < L \tag{22}$$

Singularity Functions

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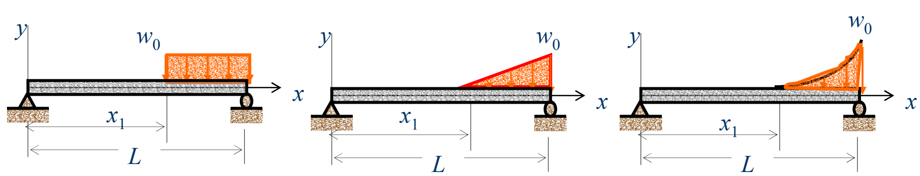
- Moment due to Distributed Loads



$$M_{w_0} = -\frac{w_0}{2}\langle x - x_1\rangle^2 \qquad M_{w_0} = -\frac{w_0}{6(L - x_1)}\langle x - x_1\rangle^3 \qquad M_{w_0} = -k\langle x - x_1\rangle^{n+2}$$

Figure 12. Open-ended-to-right distributed loads

Singularity Functions

- Moment due to Distributed Loads

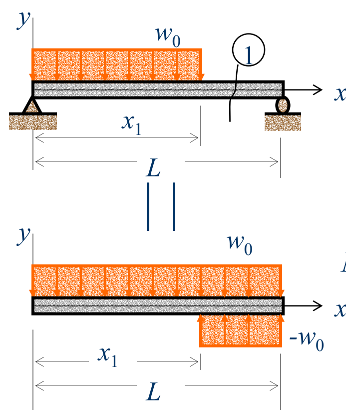


Figure 13

The moment at section 1 due to distributed load alone is

$$M_{w_0} = -\frac{w_0}{2}\langle x-0\rangle^2 + \frac{w_0}{2}\langle x-x_1\rangle^2$$

Singularity Functions

- Moment due to Distributed Loads

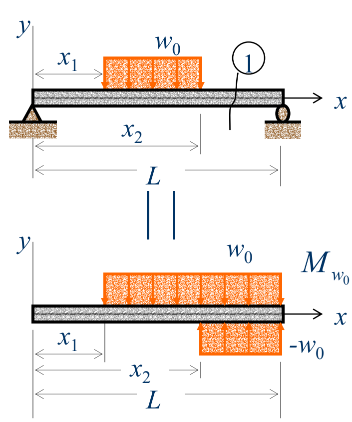


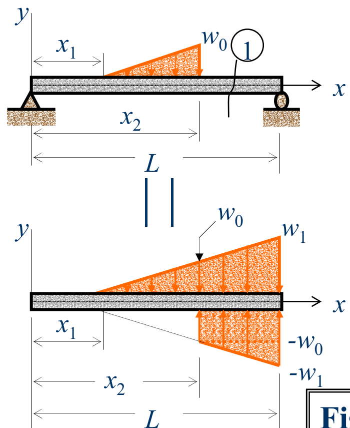
Figure 14

The moment at section 1 due to distributed load alone is

$$M_{w_0} = -\frac{w_0}{2}\langle x-x_1\rangle^2 + \frac{w_0}{2}\langle x-x_2\rangle^2$$

Singularity Functions

■ Moment due to Distributed Loads



$$\frac{w_1}{w_0} = \frac{L - x_1}{x_2 - x_1} \Rightarrow w_1 = \frac{w_0 (L - x_1)}{x_2 - x_1}$$

The moment at section 1 due to distributed load alone is

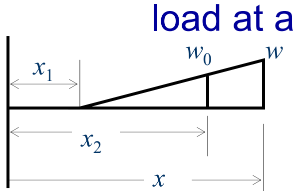
$$M_{w_0} = -\frac{w_0}{6(x_2 - x_1)} \langle x - x_1 \rangle^3 + \frac{w_0}{6(x_2 - x_1)} \langle x - x_2 \rangle^3 + \frac{w_0 \langle x - x_2 \rangle^2}{2}$$

Figure 15

Singularity Functions

■ Moment due to Distributed Loads

Note that in Fig. 14, the linearly varying load at any point $x \geq x_1$ is



$$w = \frac{w_0 (x - x_1)}{x_2 - x_1}$$

From similar triangles:

$$\frac{w}{w_0} = \frac{x - x_1}{x_2 - x_1}$$

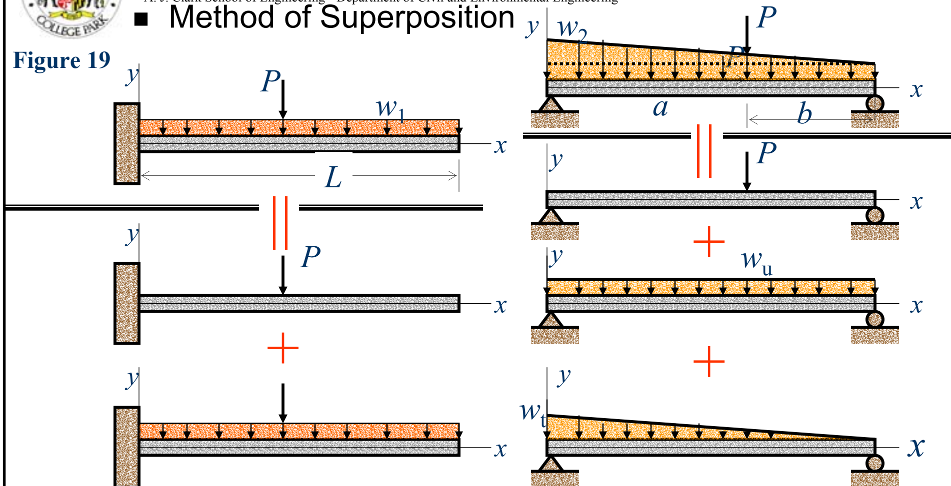
The moment of this load for Any point $x \geq x_1$ is

$$M = -\frac{1}{2}\left[\frac{w_0 (x - x_1)}{x_2 - x_1}(x - x_1)\right]\left(\frac{x - x_1}{3}\right) = \frac{w_0}{6(x_2 - x_1)}(x - x_1)^3$$

Deflection by Superposition

- Method of Superposition

Figure 19

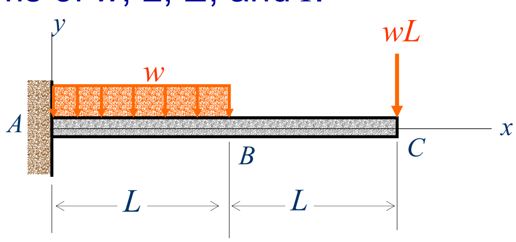


Deflection by Superposition

- Example 7

Use the method of superposition, determine the deflection at the free end of the cantilever beam shown in Fig. 27 in terms of *w*, *L*, *E*, and *I*.

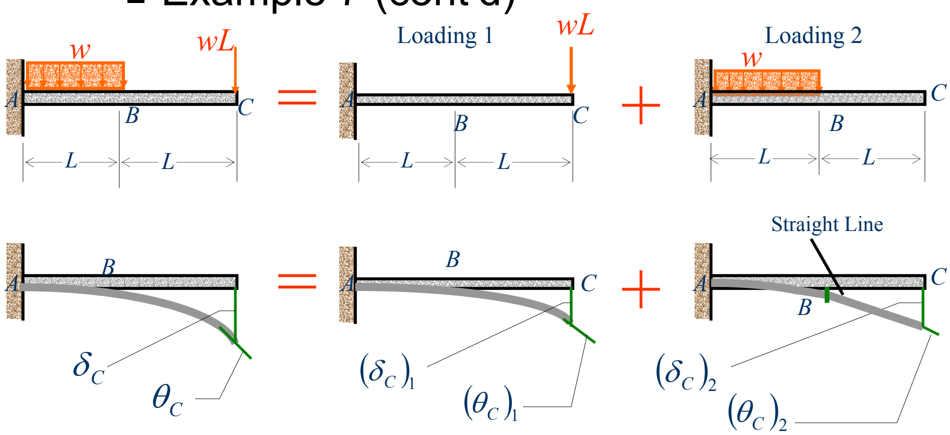
Figure 27



Deflection by Superposition

Figure 28

- Example 7 (cont'd)



Deflection by Superposition

- Example 7 (cont'd)

  Using the solutions listed in Table 1a. Cases 1 and 2 (Textbook Table B-19) with *P* = *wL*

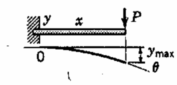
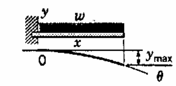
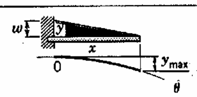
$$\delta_C = (\delta_C)_1 + (\delta_C)_2 = (\delta_C)_1 + (\delta_B)_2 + L(\theta_B)_2$$

$$= -\frac{P(2L)^3}{3EI} + \left[-\frac{wL^4}{8EI} - L\left(\frac{wL^3}{6EI}\right)\right]$$

$$= -\frac{wL(2L)^3}{3EI} + \left[-\frac{wL^4}{8EI} - L\left(\frac{wL^3}{6EI}\right)\right] = \boxed{-\frac{71wL^4}{24EI}}$$

Deflection by Superposition

- Slopes and Deflection Tables **Table 1a**

Case	Load and Support (Length L)	Slope at End (+⦤)	Maximum Deflection (+ upward)
1		$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$y_{max} = -\frac{PL^3}{3EI}$ at $x = L$
2		$\theta = -\frac{wL^3}{6EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{8EI}$ at $x = L$
3		$\theta = -\frac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{30EI}$ at $x = L$

Statically Indeterminate Beams

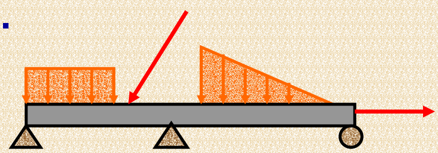
- Introduction
  - In all of the problems discussed so far, it was possible to determine the forces and stresses in beams by utilizing the equations of equilibrium, that is

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_A = 0 \qquad (29)$$

Statically Indeterminate Structures

- Statically Indeterminate Beam

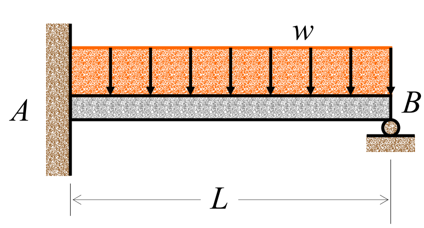
  When the equilibrium equations alone are not sufficient to determine the loads or stresses in a beam, then such beam is referred to as *statically indeterminate* beam.



Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition

  Determine the reactions at the supports for the simply supported cantilever beam (Fig.35) presented earlier for the integration method.



Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)
  - First consider the reaction at *B* as redundant and release the beam from the support (remove restraint).
  - The reaction R_B is now considered as an unknown load (see Fig. 39) and will be determined from the condition that the deflection at *B* must be zero.

Statically Indeterminate Transversely Loaded Beams

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- Illustrative Example using Superposition Method (cont'd)

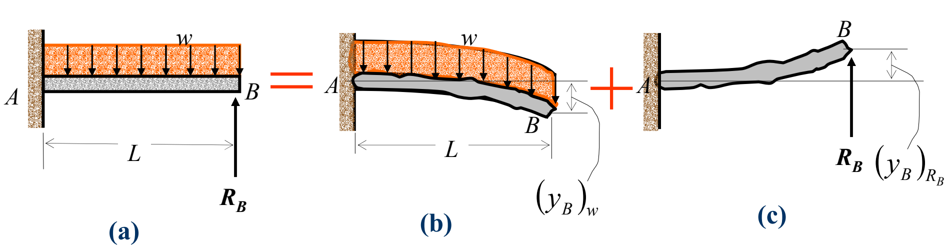

Figure 39. Original Loading is Broken into Two Loads

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

In reference to Table 1a cases 1 and 2 (Table B19 of Textbook):

$$(y_B)_{R_B} = +\frac{R_B L^3}{3EI} \quad \text{and} \quad (y_B)_w = -\frac{wL^4}{8EI} \qquad (37)$$

The deflection at *B* in the original structural configuration must equal to zero, that is

$$y_B = (y_B)_{R_A} + (y_B)_w = 0 \qquad (38)$$

Statically Indeterminate Transversely Loaded Beams

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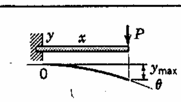
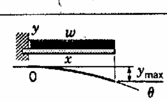
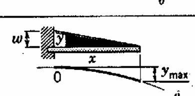
- Slopes and Deflection Tables **Table 1a**

Case	Load and Support (Length L)	Slope at End (+∠)	Maximum Deflection (+ upward)
1		$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$y_{max} = -\frac{PL^3}{3EI}$ at $x = L$
2		$\theta = -\frac{wL^3}{6EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{8EI}$ at $x = L$
3		$\theta = -\frac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{30EI}$ at $x = L$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

Substituting Eq. 37 into Eq. 38, gives

$$+\frac{R_B L^3}{3EI} - \frac{wL^4}{8EI} = 0 \qquad (39)$$

Solving for R_B, the result is

$$R_B = +\frac{3}{8}wL \qquad (40)$$

Statically Indeterminate Transversely Loaded Beams

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- Illustrative Example using Superposition Method (cont'd)

From the free-body diagram for entire beam (Figure 40), the equations of equilibrium are used to find the rest of the reactions.

$$+\uparrow \sum F_y = 0;\ R_{A_y} + R_B - wL = 0$$

$$\therefore R_{A_y} = wL - R_B \qquad (41)$$

Statically Indeterminate Transversely Loaded Beams

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- Illustrative Example using Superposition Method (cont'd)

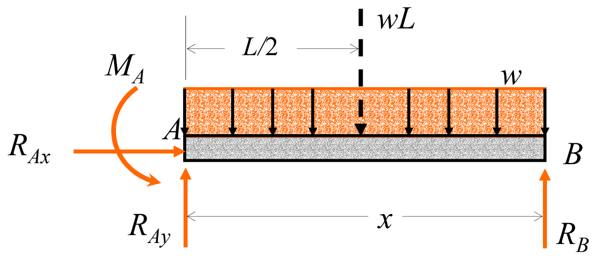

Figure 40. Free-body Diagram for the Entire Beam

Statically Indeterminate Transversely Loaded Beams

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- Illustrative Example using Superposition Method (cont'd)

But $R_B = \frac{3}{8}wL$ from Eq. 40, therefore

$$R_A = wL - \frac{3}{8}wL = \boxed{\frac{5}{8}wL} \qquad (42)$$

$$+\circlearrowleft \sum M_A = 0;\ -M_A - R_B L + (wL)\frac{L}{2} = 0$$

$$\therefore M_A = -R_B L + \frac{1}{2}wL^2 = \left(\frac{3}{8}wL\right)L - \frac{1}{2}wL^2$$

$$= \boxed{\frac{1}{8}wL^2} \qquad (43)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

From Eqs.40, 42, and 43,

$$\boxed{R_{A_y} = \frac{5}{8} wL}$$

$$\boxed{M_A = \frac{1}{8} wL^2} \qquad \boxed{R_B = \frac{3}{8} wL}$$

Which confirms the results found by using the integration method.