ENES 220 – Mechanics of Materials
Spring 2003

Solutions to Homework #9

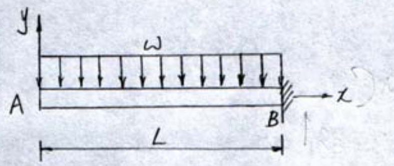
Problem 9.3

Given: the loading show

To: determine (a) the equation of the elastic curve for beam AB
(b) the deflection at the free end
(c) the slope at the free end.

FBD: as shown

FPU: Integration

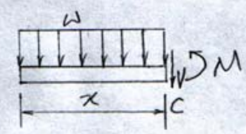


Solution:

<u>Equilibrium</u>

$\circlearrowleft + \sum M_c = 0 \qquad (\omega x)\cdot\frac{x}{2} + M = 0$

$M = -\frac{1}{2}\omega x^2$



<u>Differential equation of the elastic curve</u>

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}\omega x^2$$

$$EI\frac{dy}{dx} = -\frac{1}{6}\omega x^3 + C_1$$

$$EI\,y = -\frac{1}{24}\omega x^4 + C_1 x + C_2$$

<u>Boundary conditions</u>

$[x=L,\ y=0] \qquad 0 = -\frac{1}{24}\omega L^4 + C_1 L + C_2$

$[x=L,\ \frac{dy}{dx}=0] \qquad 0 = -\frac{1}{6}\omega L^3 + C_1$

$\therefore\ C_1 = \frac{1}{6}\omega L^3, \quad C_2 = -\frac{1}{8}\omega L^4$

$\therefore$ (a) <u>Equation of elastic curve</u>

$$y = -\frac{\omega}{24EI}\left(x^4 - 4L^3x + 3L^4\right)$$

(b) <u>Deflection at the free end</u>

$$y_A = y\big|_{x=0} = -\frac{\omega}{24EI}\cdot 3L^4 = -\frac{\omega L^4}{8EI}$$

(c) <u>the slope at the free end</u>

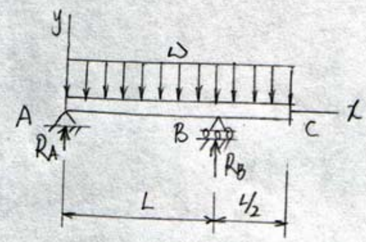
$$EI\frac{dy}{dx} = -\frac{1}{6}\omega x^3 + \frac{1}{6}\omega L^3$$

$$\therefore\ \theta_A = \frac{dy}{dx}\Big|_{x=0} = \frac{\omega L^3}{6EI}$$

Problem 9.5

Given the beam and loading shown

To determine (a) the equation of elastic curve for AB
(b) the slope at A
(c) the slope at B



FBD as shown

FPU integration

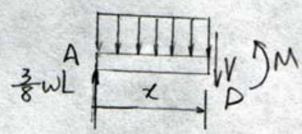
Solution:

Reactions

$$+\circlearrowright \sum M_B = 0 \quad -R_A L + (\tfrac{3}{2}wL)\cdot(\tfrac{1}{4}L) = 0$$

$$R_A = \tfrac{3}{8}wL$$

Free body diagram for portion AB



$$+\circlearrowright \sum M_D = 0 \quad -\tfrac{3}{8}wLx + (wx)\tfrac{x}{2} + M = 0$$

$$M = \tfrac{3}{8}wLx - \tfrac{1}{2}wx^2$$

Equation of elastic curve for portion AB

$$EI\frac{d^2y}{dx^2} = \tfrac{3}{8}wLx - \tfrac{1}{2}wx^2$$

$$EI\frac{dy}{dx} = \tfrac{3}{16}wLx^2 - \tfrac{1}{6}wx^3 + C_1$$

$$EI\,y = \tfrac{1}{16}wLx^3 - \tfrac{1}{24}wx^4 + C_1x + C_2$$

Boundary conditions

$$[x=0,\ y=0] \quad 0 = 0 - 0 + 0 + C_2 \quad \therefore C_2 = 0$$

$$[x=L,\ y=0] \quad 0 = \tfrac{1}{16}wL\cdot L^3 - \tfrac{1}{24}wL^4 + C_1L \quad \therefore C_1 = -\tfrac{1}{48}wL^3$$

(a) Elastic curve

$$y = \frac{w}{EI}\left(\tfrac{1}{16}Lx^3 - \tfrac{1}{24}x^4 - \tfrac{1}{48}L^3x\right)$$

$$\frac{dy}{dx} = \frac{w}{EI}\left(\tfrac{3}{16}Lx^2 - \tfrac{1}{6}x^3 - \tfrac{1}{48}L^3\right)$$

(b) the slope at A

$$\theta_A = \frac{dy}{dx}\Big|_{x=0} = \frac{w}{EI}\left(0 - 0 - \tfrac{1}{48}L^3\right) = -\frac{wL^3}{48EI}$$

(c) the slope at B

$$\theta_B = \frac{dy}{dx}\Big|_{x=L} = \frac{w}{EI}\left(\tfrac{3}{16}L^3 - \tfrac{1}{6}L^3 - \tfrac{1}{48}L^3\right) = 0$$

Problem 9.12

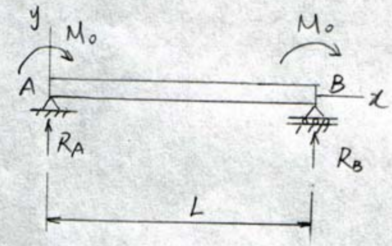
Given: the beam and loading shown. Beam AB is W460X113
$M_0 = 224\ \mathrm{kN\cdot m}$ $E = 200\ \mathrm{GPa}$

To: determine (a) the location and magnitude of the absolute deflection
(b) If the maximum deflection $\le 12\ \mathrm{mm}$, the maximum allowable length L.

FBD: as shown

FPU: integration

Solution

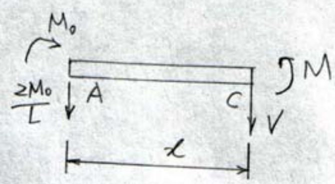


Reactions: Using AB as a free body

$+\circlearrowleft \Sigma M_B = 0 \quad -2M_0 - R_A L = 0$

$\therefore R_A = -\frac{2M_0}{L}$

Free body diagram of AC



$+\circlearrowleft \Sigma M_C = 0 \quad -M_0 + \frac{2M_0}{L}x + M = 0$

$M = \frac{M_0}{L}(L - 2x)$

Equation of elastic curve

$EI\frac{d^2y}{dx^2} = \frac{M_0}{L}(L-2x)$

$EI\frac{dy}{dx} = \frac{M_0}{L}(Lx - x^2) + C_1$

$EI\,y = \frac{M_0}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1 x + C_2$

Boundary Conditions

$[x=0,\ y=0] \quad 0 = 0 - 0 + 0 + C_2 \quad \therefore C_2 = 0$

$[x=L,\ y=0] \quad 0 = \frac{M}{L}\left(\frac{L^3}{2} - \frac{L^3}{3}\right) + C_1 L \quad C_1 = -\frac{1}{6}M_0 L$

$\therefore y = \frac{M_0}{EIL}\left(\frac{1}{2}Lx^2 - \frac{1}{3}x^3 - \frac{1}{6}L^2 x\right)$

$\frac{dy}{dx} = \frac{M_0}{EI}\left(Lx - x^2 - \frac{1}{6}L^2\right)$

(a) Location and magnitude of maximum deflection

$\frac{dy}{dx} = \frac{M_0}{EI}\left(Lx - x^2 - \frac{1}{6}L^2\right) = 0 \quad \therefore x_m = 0.21132L$

$y_m = \frac{M_0}{EIL}\left(\frac{1}{2}L(0.21132L)^2 - \frac{1}{3}(0.21132L)^3 - \frac{1}{6}L^2(0.21132L)\right)$

$= -0.0160375\frac{M_0 L^2}{EI}$

$|y_m| = 0.0160375\frac{M_0 L^2}{EI}$

(b) Maximum allowable length L

$L = \left\{\frac{EI|y_m|}{0.0160375 M_0}\right\}^{\frac{1}{2}} = \left\{\frac{(200\times10^9 \mathrm{Pa})(556\times10^{-6}\mathrm{m^4})(1.2\times10^{-3}\mathrm{m})}{0.0160375\cdot(224\times10^3\ \mathrm{N\cdot m})}\right\}^{\frac{1}{2}}$

$= 6.09\ \mathrm{m}$

Problem 9.14

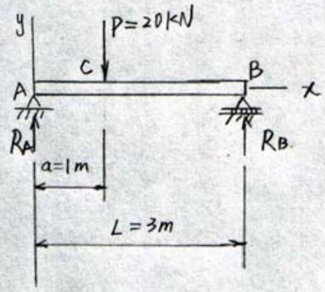
Given the beam and loading shown. $E = 200\,GPa$, the beam is W150X18.0

To determine the deflection at C

FBD as shown.

FPU Integration

Solution:



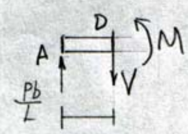
<u>Reactions</u> Using AB as a free body

Let $b = L - a$ $\quad b = 3m - 1m = 2m$

$+\circlearrowleft M_B = 0 \quad -R_A \cdot L + P \cdot b = 0 \quad \therefore R_A = \frac{Pb}{L}$

$+\circlearrowleft M_A = 0 \quad R_B \cdot L - P \cdot a = 0 \quad R_B = \frac{Pa}{L}$

<u>Equation of elastic curve for AC</u> $(0 < x < a)$



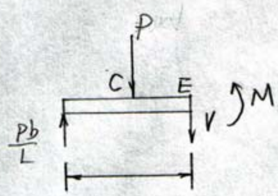
$+\circlearrowleft M_D = 0 \quad -\frac{Pb}{L} \cdot x + M = 0$

$\therefore EI\frac{d^2y}{dx^2} = M = \frac{Pb}{L}x$

$EI\frac{dy}{dx} = \frac{Pb}{L}(\frac{1}{2}bx^2) + C_1 \quad (1)$

$EI\,y = \frac{Pb}{L}(\frac{1}{6}bx^3) + C_1x + C_2 \quad (2)$

<u>Equation of elastic curve for CB</u> $(a < x < L)$



$+\circlearrowleft M_E = 0 \quad -\frac{Pb}{L} \cdot x + P \cdot (x-a) + M = 0$

$EI\frac{d^2y}{dx^2} = M = \frac{P}{L}[bx - L(x-a)]$

$EI\frac{dy}{dx} = \frac{P}{L}[\frac{1}{2}bx^2 - \frac{1}{2}L(x-a)^2] + C_3 \quad (3)$

$EI\,y = \frac{P}{L}[\frac{1}{6}bx^3 - \frac{1}{6}L(x-a)^3] + C_3x + C_4 \quad (4)$

<u>Boundary conditions</u>

$[x=0,\ y=0]$ from (2), $\quad 0 = 0 + 0 + C_2 \quad \therefore C_2 = 0$

$[x=a \quad \frac{dy}{dx} = \frac{dy}{dx}]$ Eqs (1) and (3) $\quad \frac{Pb}{L}(\frac{1}{2}ba^2) + C_1 = \frac{Pb}{L}(\frac{1}{2}ba^2) + C_3 \quad \therefore C_1 = C_3$

$[x=a \quad y=y]$ Eqs (2) and (4) $\quad \frac{Pb}{L}(\frac{1}{6}ba^3) + C_1a = \frac{P}{L}[(\frac{1}{6}ba^3) + 0] + C_1a + C_4 \quad \therefore C_4 = 0$

$[x=L,\ y=0]$ from (4) $\quad \frac{P}{L}[\frac{1}{6}bL^3 - \frac{1}{6}L(L-a)^3] + C_3L = 0$

$\therefore C_1 = C_3 = \frac{P}{L}(\frac{1}{6}b^3 - \frac{1}{6}bL^2)$

<u>Deflection at C</u>

Make $x = a$ in eq (2)

$y_c = \frac{P}{EIL}[\frac{1}{6}ba^3 + \frac{1}{6}b^3a - \frac{1}{6}bL^2a] = \frac{P}{6EIL}(ba^3 + b^3a - L^2ab)$

$\because$ W150X18.0 $\quad \therefore I = 9.17 \times 10^6\,mm^4 = 9.17 \times 10^{-6}\,m^4$

$\therefore y_c = \frac{20 \times 10^3\,N}{(200 \times 10^9\,Pa)(9.17 \times 10^{-6}\,m^4)(3m)}\left((2m)(1m)^3 + (2m)^3(1m) - (3m)^2(1m)(2m)\right)$

$= -4.85 \times 10^{-3}\,m = -4.85\,mm$

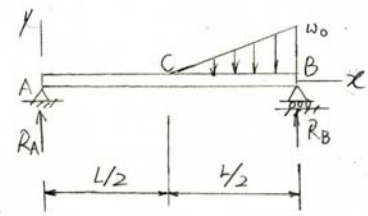
Problem 9.38

Given the beam and loading shown

To determine (a) equation of the elastic curve
(b) the slope at end A
(c) the deflection of point C

FBD as shown

FPU singularity function



$[x=0, y=0]$ $[x=L, y=0]$

$[x=0, M=0]$ $[x=L, M=0]$

Solution:

Equilibrium

$+\uparrow \Sigma F_y = 0 \quad R_A - \frac{w_0}{2}\cdot\frac{L}{2} + R_B = 0$

$\circlearrowleft \Sigma M_B = 0 \quad -R_A \cdot L + \frac{w_0}{2}\cdot\frac{L}{2}\cdot\frac{L}{6} = 0$

$\therefore \quad R_A = \frac{1}{24} w_0 L \quad R_B = \frac{5}{24} w_0 L$

Bending moment

$w = \frac{2w_0}{L}\langle x - \frac{L}{2}\rangle^1$

$\frac{dV}{dx} = -w = -\frac{2w_0}{L}\langle x - \frac{L}{2}\rangle^1$

$\frac{dM}{dx} = V = \frac{1}{24}w_0 L - \frac{2w_0}{L}\langle x - \frac{L}{2}\rangle^2$

$M = -\frac{1}{3}\frac{w_0}{L}\langle x - \frac{L}{2}\rangle^3 + \frac{1}{24}w_0 L\, x$

a) Equation of the elastic curve

$EI\frac{d^2y}{dx^2} = -\frac{1}{3}\frac{w_0}{L}\langle x - \frac{L}{2}\rangle^3 + \frac{1}{24}w_0 L x$

$EI\frac{dy}{dx} = -\frac{1}{12}\frac{w_0}{L}\langle x - \frac{L}{2}\rangle^4 + \frac{1}{48}w_0 L x^2 + c_1$

$EI\, y = -\frac{1}{60}\frac{w_0}{L}\langle x - \frac{L}{2}\rangle^5 + \frac{1}{144}w_0 L x^3 + c_1 x + c_2$

Boundary conditions

$[x=0, y=0] \quad 0 = 0 + 0 + 0 + c_2 \quad \therefore c_2 = 0$

$[x=L, y=0] \quad -\frac{1}{60}\frac{w_0}{L}\left(\frac{L}{2}\right)^5 + \frac{1}{144}w_0 L^4 + c_1 L + 0 = 0$

$\therefore c_1 = -\frac{37}{5760} w_0 L^3$

$\therefore y = \frac{w_0}{EIL}\left\{-\frac{1}{60}\langle x - \frac{L}{2}\rangle^5 + \frac{1}{144}L^2 x^3 - \frac{37}{5760}L^4 x\right\}$

(b) Slope at A

$\frac{dy}{dx} = \frac{w_0}{EIL}\left\{-\frac{1}{12}\langle x - \frac{L}{2}\rangle^4 + \frac{1}{48}w_0 L^2 x^2 - \frac{37}{5760}L^4\right\}$

$\theta_A = \left.\frac{dy}{dx}\right|_{x=0} = \frac{w_0}{EIL}\left\{0 + 0 - \frac{37}{5760}L^4\right\}$

$= -\frac{37}{5760}\frac{w_0 L^3}{EI}$

(c) Deflection at C

$y_c = y|_{x=\frac{L}{2}}$

$= \frac{w_0}{EIL}\left\{0 + \frac{1}{144}L^2\left(\frac{L}{2}\right)^2 - \frac{37}{5760}L^4\left(\frac{L}{2}\right)\right\}$

$= -\frac{3}{1280}\frac{w_0 L^4}{EI}$

Problem 9.40

Given the beam and the loading shown

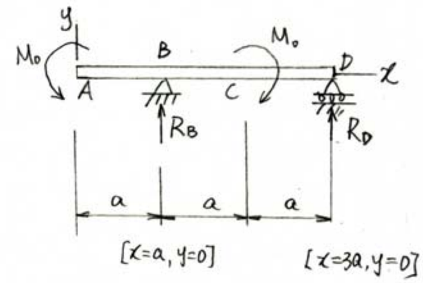
To determine (a) the deflection at end A

(b) the deflection at point C

(c) the slope at end D

FBD as shown

FPU singularity function



Solution:

Equilibrium

Since loads self equilibrate

$R_B = R_D = 0$

$0 < x < 2a \qquad M = -M_0$

$2a < x < 3a \qquad M = 0$

Singularity function

$M = -M_0 + M\langle x-2a\rangle^0$

Equation of elastic curve

$EI\frac{dy^2}{dx^2} = M = -M_0 + M_0\langle x-2a\rangle^0$

$EI\frac{dy}{dx} = -M_0 x + M_0\langle x-2a\rangle^1 + C_1$

$EI\,y = -\frac{1}{2}M_0 x^2 + \frac{1}{2}M_0\langle x-2a\rangle^2 + C_1 x + C_2$

Boundary conditions.

$[x=a,\ y=0] \quad -\frac{1}{2}M_0 a^2 + 0 + C_1 a + C_2 = 0$

$aC_1 + C_2 = \frac{1}{2}M_0 a^2$

$[x=3a,\ y=0] \quad -\frac{1}{2}M_0(3a)^2 + \frac{1}{2}M_0 a^2 + C_1(3a) + C_2 = 0$

$3aC_1 + C_2 = 4M_0 a^2$

$\therefore C_1 = \frac{7}{4}M_0 a \qquad C_2 = -\frac{5}{4}M_0 a^2$

$\therefore y = \frac{M_0}{EI}\left\{-\frac{1}{2}x^2 + \frac{1}{2}\langle x-2a\rangle^2 + \frac{7}{4}ax - \frac{5}{4}a^2\right\}$

$\frac{dy}{dx} = \frac{M_0}{EI}\left\{-x + \langle x-2a\rangle^1 + \frac{7}{4}a\right\}$

(a) Deflection at A

$y_A = y|_{x=a} = \frac{M_0 a^2}{EI}\left\{0 + 0 + 0 - \frac{5}{4}\right\}$

$= -\frac{5M_0 a^2}{4EI}$

(b) Deflection at C

$y_C = y|_{x=2a}$

$= \frac{M_0 a^2}{EI}\left\{-\frac{1}{2}(2^2) + 0 + \frac{7}{4}(2) - \frac{5}{4}\right\}$

$= \frac{M_0 a^2}{4EI}$

(c) Slope at D

$\theta_D = \frac{dy}{dx}\Big|_{x=3a}$

$= \frac{M_0 a}{EI}\left\{-3 + 1 + \frac{7}{4}\right\}$

$= -\frac{M_0 a}{4EI}$

Problem 9.43

Given the beam and loading shown

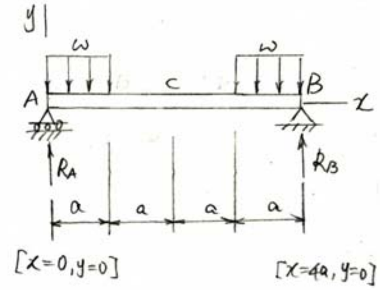
To determine (a) the equation of elastic curve

(b) the deflection at the midpoint C

FBD as shown

FPU singularity function

Solution



Equilibrium

By symmetry, $R_A = R_B$

$+\uparrow \Sigma F_y = 0 \quad R_A + R_B - 2wa = 0$

$\therefore \quad R_A = R_B = wa$

Equation of elastic curve

$$w(x) = w - w\langle x-a\rangle^0 + w\langle x-3a\rangle^0$$

$$\frac{dv}{dx} = -w(x) = -w + w\langle x-a\rangle^0 - w\langle x-3a\rangle^0$$

$$\frac{dM}{dx} = V = wa - wx + w\langle x-a\rangle^1 - w\langle x-3a\rangle^1$$

$$EI\frac{d^2y}{dx^2} = M = wax - \frac{1}{2}wx^2 + \frac{1}{2}w\langle x-a\rangle^2 - \frac{1}{2}w\langle x-3a\rangle^2$$

$$EI\frac{dy}{dx} = \frac{1}{2}wax^2 - \frac{1}{6}wx^3 + \frac{1}{6}w\langle x-a\rangle^3 - \frac{1}{6}w\langle x-3a\rangle^3 + C_1$$

$$EIy = \frac{1}{6}wax^3 - \frac{1}{24}wx^4 + \frac{1}{24}w\langle x-a\rangle^4 - \frac{1}{24}w\langle x-3a\rangle^4 + C_1x + C_2$$

Boundry Conditions

$[x=0, y=0] \quad 0 - 0 + 0 - 0 + 0 + C_2 = 0 \qquad \therefore C_2 = 0$

$[x=4a, y=0] \quad \frac{1}{6}wa(4a)^3 - \frac{1}{24}w(4a)^4 + \frac{1}{24}w(3a)^4 - \frac{1}{24}wa^4 + C_1(4a) = 0$

$\therefore \quad C_1 = -\frac{5}{6}wa^3$

(a) Equation of elastic curve

$$y = \frac{w}{EI}\left\{\frac{1}{6}wax^3 - \frac{1}{24}wx^4 + \frac{1}{24}w\langle x-a\rangle^4 - \frac{1}{24}w\langle x-3a\rangle^4 - \frac{5}{6}a^3x\right\}$$

(b) Deflection at C

$$y_c = y\Big|_{x=2a} = \frac{wa^4}{EI}\left\{\frac{1}{6}(2)^3 - \frac{1}{24}(2)^4 + \frac{1}{24}(1)^4 - \frac{5}{6}(2)\right\}$$

$$= -\frac{23wa^4}{24EI}$$

Problem 9.47

Given the timber beam and loading shown. $E = 1.6 \times 10^6$ psi

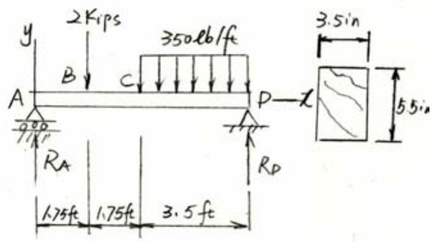
To determine (a) the slope at end A

(b) the deflection at the midpoint C

FBD as shown

FPU singularity function

Solution:



Equilibrium

$\circlearrowleft \Sigma M_D = 0 \quad (-7\,ft) R_A + (5.25\,ft)(2\,Kips)$
$+ (1.75\,ft)(0.35\,Kips/ft)(3.5\,ft) = 0$

$R_A = 1.80625$ Kips

Equation of the elastic curve

$w(x) = 0.35 \langle x-3.5 \rangle^0$ Kip/ft

$\frac{dV}{dx} = -w = -0.35 \langle x-3.5 \rangle^0$ Kip/ft

$\frac{dM}{dx} = V = 1.80625 - 2\langle x-1.75 \rangle^0 - 0.35 \langle x-3.5 \rangle^1$ Kip

$EI \frac{d^2y}{dx^2} = M = 1.80625x - 2\langle x-1.75 \rangle^1 - 0.175 \langle x-3.5 \rangle^2$ Kip·ft

$EI \frac{dy}{dx} = 0.903125\, x^2 - \langle x-1.75 \rangle^2 - 0.05833 \langle x-3.5 \rangle^3 + C_1$ Kip·ft^2

$EI \frac{dy}{dx} = 0.301042\, x^3 - \frac{1}{3} \langle x-1.75 \rangle^3 - 0.014583 \langle x-3.5 \rangle^4 + C_1 x + C_2$ Kip·ft^3

Boundary conditions

$[x=0, y=0] \quad C_2 = 0$

$[x = 7\,ft, y=0] \quad 0.301042(7)^3 - \frac{1}{3}(5.25)^3 - 0.014583(3.5)^4 + C_1(7) + 0 = 0$

$C_1 = -7.54779$ Kip·ft^2

$I = \frac{1}{12} b h^3 = \frac{1}{12}(3.5\,in)(5.5\,in)^3 = 48.526\,in^3$

$EI = (1.6 \times 10^6 psi)(48.526\,in^3) = 77.6417$ kip·in^2 $= 539.18$ kip·ft^2

(a) Slope at A

$EI \frac{dy}{dx}\Big|_{x=0} = 0 - 0 - 0 - 7.54779$ Kip·ft^2

$\theta_A = -\frac{7.54779\ Kip \cdot ft^2}{539.18\ Kip \cdot ft^2} = -13.99 \times 10^{-3}$ rad

(b) Deflection at C

$EI\, y\big|_{x=3.5ft} = 0.301042(3.5)^3 - \frac{1}{3}(1.5)^3 - 0 - (7.54779)(3.5) + 0$

$= -15.297$ Kip·ft^3

$y_c = -\frac{15.297\ Kip \cdot ft^3}{539.18\ Kip \cdot ft^3} = -28.37 \times 10^{-3}$ ft