CHAPTER

6b

Risk Analysis in Engineering and Economics

ENGINEERING ECONOMICS AND FINANCE

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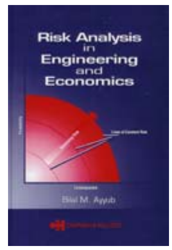

Risk Analysis for Engineering

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Economic Equivalence Involving Interest

- The Meaning of Equivalence
  - Economic equivalence is used commonly in engineering to compare alternatives.
  - In engineering economy, two things are said to be equivalent if they have the same effect.
  - Unlike most individuals involved with personal finances, corporate and government decision makers using engineering economics might not be so much concerned with the timing of a project's cash flows as with the profitability of the project.

Economic Equivalence Involving Interest

- The Meaning of Equivalence (cont'd)
  - Therefore, analytical tools are needed to compare projects involving receipts and disbursements occurring at different times, with the goal of identifying an alternative having the largest eventual profitability.

Economic Equivalence Involving Interest

- Equivalence Calculations
  - Several equivalence calculations are presented in this section, where these calculations involve the following:
    1. cash flows,
    2. interest rates,
    3. bond prices, and
    4. loans

Economic Equivalence Involving Interest

- Equivalence Calculations (cont'd)
  - Two cash flows need to be presented along the same time period using a similar format to facilitate comparison.
  - When interest is earned, monetary amounts can be directly added only if they occur at the same point in time.
  - Equivalent cash flows are those that have the same value.

Economic Equivalence Involving Interest

- Equivalence Calculations (cont'd)
  - For loans, the effective interest rate for the loan, called also the internal rate of return, is defined as the rate that sets the receipts equal to the disbursements on an equivalent basis.
  - The equivalence of two cash flows can be assessed at any point in time as illustrated in Example 19.

Economic Equivalence Involving Interest

- **Example 19:** Equivalence Between Cash Flows
  - Two equivalent cash flows are presented in Table 5.
  - The equivalence can be established at any point in time for an example interest rate of 12% compounded annually.
  - For example, if eight years were selected,

$$F = \$1{,}000(1+0.12)^{8} = \$2{,}475.96$$

for cash flow 1.

Economic Equivalence Involving Interest

- Example 19 (cont'd):

Table 5. Two Equivalent Cash Flows

Year	Cash Flow 1	Cash Flow 2
1	$1,000.00	$0.00
2	$0.00	$0.00
3	$0.00	$0.00
4	$0.00	$1,000.00
5	—	—
6	—	—
7	—	—
8	$2475.96	$1,573.50

Economic Equivalence Involving Interest

- Example 19 (cont'd):
  - While $F = \$1{,}000(1 + 0.12)^4 = \$1{,}573.50$ for cash flow 2.
  - It should be noted that two or more distinct cash flows are equivalent if they result into the same amount at the same point in time.
  - In this case, the two cash flows are not equivalent.

Economic Equivalence Involving Interest

- **Example 20:** Internal Rate of Return
  - According to the equivalence principle, the actual interest rate earned on an investment can be defined as the interest rate that sets the equivalent receipts to the equivalent disbursements.
  - For Table 6, the following equality can be set as:

$$\begin{aligned}&\$1{,}000+\$500(P/F,i,1) + \$250(P/F,i,5)\\ &= \$482(P/A,i,3)(P/F,i,1) + \$482(P/A,i,2)(P/F,i,5) \qquad (39)\end{aligned}$$

Economic Equivalence Involving Interest

- Example 20 (cont'd):

Table 6. Converting Cash Flow to its Present Value

Time (Year End)	Receipts ($)	Disbursements ($)
0	0.00	-1000.00
1	0.00	-500.00
2	482.00	0.00
3	482.00	0.00
4	482.00	0.00
5	0.00	-250.00
6	482.00	0.00
7	482.00	0.00

Economic Equivalence Involving Interest

- Example 20 (cont'd):
  - By trial and error i = 10% makes the above equation valid.
  - The equivalence can be made at any point of reference in time; it does not need to be the origin (time = zero) to produce the same answer.
  - If the receipts and disbursement of an investment cash flow are equivalent for some interest rate, the cash flows of any two portions of the investment have equal absolute equivalent values at that interest rate.

Economic Equivalence Involving Interest

- Example 20 (cont'd):
  - That is, the negative (-) of the equivalent amount of one cash flow portion is equal to the equivalent of the remaining portion on the investment.
  - breaking up the above cash flow between years 4 and 5, and performing the equivalence at the 4th year produces the following:

$$-\$1{,}000(F/P,10,4)-\$500(F/P,10,3)+\$482(F/A,10,3) = -(-\$250(P/F,10,1)+\$482(P/A,10,2)(P/F,10,1))$$
$$-\$1{,}000(1.464)-\$500(1.331)+\$482(3.310) = -(-\$250(0.9091)+\$482(1.7355)(0.9091))$$
$$-\$534 = -\$534 \tag{40}$$

Economic Equivalence Involving Interest

- **Example 21:** Bond Prices
  - A bond is bought for \$900 and has a face value of \$1,000 with 6% annual interest that is paid semiannually.
  - The bond matures in 7 years.
  - The yield to maturity is defined as the rate of return on the investment for its duration.

Economic Equivalence Involving Interest

- Example 21 (cont'd):
  - Using equivalence, the following equality can be developed:

$$\$900 = \$30(P/A,i,14) + \$1{,}000(P/F,i,14) \quad (41)$$

  - By trial and error i = 3.94% per semiannual period.
  - The nominal rate is 2(3.94) = 7.88%, while the effective rate is 8.04%.

Economic Equivalence Involving Interest

- **Example 22:** Equivalence Calculations for Loans
  - Suppose a five-year loan of $10,000 (with interest of 16% compounded quarterly with quarterly payments) is to be paid off after the 13th payment. The quarterly payment is

$$\$10{,}000(A/P,4,20) = \$10{,}000(0.0736) = \$736 \quad (42)$$

  - The balance can be based on the remaining payments as

$$\$736(P/A,4,7) = \$736(6.0021) = \$4{,}418 \quad (43)$$

Economic Equivalence Involving Interest

- Amortization Schedule for Loans
  - An amortization schedule for a loan is defined as a breakdown of each loan payment (A) into two portions of an interest payment (I_t) and a payment towards the principal balance (B_t).
  - The following terms are defined:

I_t = interest payment of A at time t,
B_t = portion of payment of A to reduce balance at time t

Economic Equivalence Involving Interest

- Amortization Schedule for Loans
  - The payment can be expressed as

$$A = I_t + B_t \quad \text{for t} = 1, 2, \ldots, n \qquad (44)$$

  - The balance at end of t-1 is given by

$$Bt = A(P/A, i, n\text{-}(t\text{-}1)) \qquad (45)$$

  - Therefore, the following relationships can be obtained

$$Bt = A(P/F, i, n\text{-}t\text{+}1) \qquad (49)$$

Economic Equivalence Involving Interest

- **Example 23:** Principal and Interest Payments
  - Suppose a four-year loan of $1,000 (with interest of 15% compounded annually with annually payments) is to be paid off.
  - The payment is A = $1,000(A/P,15,4) = $1,000(0.3503) = $350.265.
  - The results are illustrated in Table 7 based on Eq. 49 and using $I_t = A - B_t$.

Economic Equivalence Involving Interest

- Example 23 (cont'd):

Table 7. Calculations for Example 23

Year End	Loan Payment ($)	Payment towards Principal (B_t)	Interest Payment (I_t)
1	$350.265	$350.265($P/F$, 15, 4) = $200.27	$150.00
2	$350.265	$350.265($P/F$, 15, 3) = $230.30	$119.97
3	$350.265	$350.265($P/F$, 15, 2) = $264.85	$85.42
4	$350.265	$350.265($P/F$, 15, 1) = $304.58	$45.69
Total	$1,401.06	$1,000.00	$401.06

Economic Equivalence Involving Inflation

- Price Indexes
  - For purposes of calculating the effect of inflation on equivalence, price indexes are used.
  - A price index is defined as the ratio between the current price of a commodity or service to the price at some earlier reference time.

Economic Equivalence Involving Inflation

- **Example 24:** Economic Equivalence Involving Inflation
  - The base year is 1967, with an index of 100, and the commodity price is $1.46/lb.
  - If the price in 1993 is $5.74/lb, the 1993 index is $5.74/1.4 = $393.20.
  - The actual consumer price index (CPI) and annual inflation rates are published and can be used for these computations.

Economic Equivalence Involving Inflation

- Annual Inflation Rate
  - The annual inflation rate at t + 1 can be computed as:

$$\text{Annual inflation rate at } t+1 = \frac{\text{CPI}_{t+1} - \text{CPI}_t}{\text{CPI}_t} \qquad (50)$$

  - The average inflation rate $\bar{f}$ can be computed based on the following condition

$$\text{CPI}_t(1+\bar{f})^n = \text{CPI}_{t+n} \qquad (51)$$

Economic Equivalence Involving Inflation

- Annual Inflation Rate (cont'd)
  - Therefore, the average inflation rate is

$$\bar{f} = \sqrt[n]{\frac{\text{CPI}_{t+n}}{\text{CPI}_t}} - 1 \qquad (52)$$

Economic Equivalence Involving Inflation

- **Example 25:** Annual Inflation Rate
  - Assuming the CPI (of 1966) = 97.2 and the CPI (of 1980) = 246.80, the average rate of inflation over the 14-year interval can be obtained by applying Eq. 52 as follows:

$$\bar{f} = \sqrt[14]{\frac{246.80}{97.2}} - 1 = \left(\frac{246.80}{97.2}\right)^{1/14} - 1 = 6.882\% \qquad (53)$$

Economic Equivalence Involving Inflation

- Purchasing Power of Money
  - The purchasing power at time t in reference to time period $t-n$ is defined as

$$\text{Purchasing power at time } t = \frac{\text{CPI}_{t-n}}{\text{CPI}_t} \qquad (54)$$

  - Denoting the annual rate of loss in purchasing power as k, the average rate of loss of purchasing power $\bar{k}$ can be computed as:

$$\frac{\text{CPI}_{base\ year}}{\text{CPI}_t}(1-\bar{k})^n = \frac{\text{CPI}_{base\ year}}{\text{CPI}_{t+n}} \qquad (55)$$

Economic Equivalence Involving Inflation

- Purchasing Power of Money (cont'd)
  - Solving for CPIt produces the following:

$$CPI_t = (1-\bar{k})^n CPI_{t+n} \qquad (56)$$

  - Therefore,

$$(1+\bar{f})^n = \frac{1}{(1-\bar{k})^n} \qquad (57)$$

  - Equation 57 relates the average inflation rate $\bar{f}$ and the annual rate of loss in purchasing power $\bar{k}$.

Economic Equivalence Involving Inflation

- Constant Dollars
  - By definition, the constant dollar is

$$\text{Constant Dollars} = \frac{1}{(1+\bar{f})^n}(\text{Actual Dollars}) \qquad (58)$$

  - When using actual dollars, the market interest rate (i) is used.
  - When using constant dollars, use the inflation-free interest rate (i*).

Economic Equivalence Involving Inflation

- Constant Dollars (cont'd)
  - The inflation-free interest rate (i^*) is defined as follows for one year:

$$i^* = \frac{1+i}{1+f} - 1 \qquad (59)$$

  - For multiple years, it is defined as

$$i^* = \frac{1+i}{(1+f)^n} - 1 \qquad (60)$$

Economic Analysis of Alternatives

- Present, Annual, and Future-Worth Amounts
  - The present-worth amount is the difference between the equivalent receipts and disbursements at the present.
  - Assuming F_t to be a net cash flow at time t, the present worth (PW) is

$$PW(i) = \sum_{t=0}^{n} F_t(P/F,i,t) = \sum_{t=0}^{n} F_t(1+i)^{-t} \qquad (61)$$

Economic Analysis of Alternatives

- Present, Annual, and Future-Worth Amounts (cont'd)
  - The net cash flow F_t is defined as the sum of all disbursements and receipts at time t.
  - The annual equivalent amount is the annual equivalent receipts minus the annual equivalent disbursements of a cash flow.
  - It is used for repeated cash flows per year. It is calculated by applying the following equation:

$$AE(i) = PW(i)(A/P,i,n) = \left(\sum_{t=0}^{n} F_t (1+i)^{-t}\right)\left(\frac{i(1+i)^n}{(1+i)^n - 1}\right) \qquad (62)$$

Economic Analysis of Alternatives

- Present, Annual, and Future-Worth Amounts (cont'd)
  - The future worth amount is

$$FW(i) = \sum_{t=0}^{n} F_t (F/P,i,n-t) = \sum_{t=0}^{n} F_t (1+i)^{n-t} \qquad (63)$$

  - The amounts PW, AE, and FW differ in the point of time used to compare the equivalent amounts.

Economic Analysis of Alternatives

- **Example 26:** Annual Equivalent Amount
  - The cash flow illustrated in Table 8 is used to compute the annual equivalent amount based on an interest rate of 10% for a segment of the cash flow that repeats as follows:

$$AE(10) = [-\$1,000 + \$400(P/F,10,1) + \$900(P/F,10,2)](A/P,10,2) \quad (64)$$

  - or

$$AE(10) = [-\$1,000 + \$400(0.9091) + \$900(0.8265)](0.5762) = \$61.93 \quad (65)$$

Economic Analysis of Alternatives

- Example 26 (cont'd):

Table 8. Cash Flow for Example 26

Year End	Receipts ($)	Disbursements ($)
0	0.00	-1,000.00
1	400.00	0.00
2	900.00	-1,000.00
3	400.00	0.00
4	900.00	-1,000.00
⋮	⋮	⋮
n-2	900.00	-1,000.00
n-1	400.00	0.00
n	900.00	0.00

Economic Analysis of Alternatives

- Internal Rate of Return
  - The internal rate of return (IRR) is the interest rate that causes the equivalent receipts of a cash flow to be equal to the equivalent disbursements of the cash flow.
  - Solving for i^* such that the following condition is satisfied:

$$0 = PW(i^*) = \sum_{t=0}^{n} F_t (1+i^*)^{-t} \qquad (66)$$

Economic Analysis of Alternatives

- Internal Rate of Return (cont'd)
  - It represents the rate of return on the unrecovered balance of an investment (or loan).
  - The following equation can be developed for loans:

$$U_t = U_{t-1}(1+i^*) + F_t \qquad (67)$$

  - where U_0 is the initial amount of loan or first cost of an asset (F_0), F_t is the amount received at the end of the period t, and i^* is IRR.

Economic Analysis of Alternatives

- **Example 27:** Internal Rate of Return
  - The cash flow illustrated in Table 9 is used to solve for i by trial and error using the net cash flow and Eq. 66.
  - The internal rate of return was determined to be i^* = 12.8%.

Economic Analysis of Alternatives

- Example 27 (cont'd)

Table 9. Cash Flow for Example 27

Year End	Receipts ($)	Disbursements ($)
0	0.00	-1000.00
1	0.00	-800.00
2	500.00	0.00
3	500.00	0.00
4	500.00	0.00
5	1200.00	0.00

Economic Analysis of Alternatives

- Payback Period
  - The payback period without interest is the length of time required to recover the first cost of an investment from the cash flow produced by the investment for an interest rate of zero.
  - It can be computed as the smallest n that produces:

$$\sum_{t=0}^{n} F_t \geq 0 \qquad (68)$$

Economic Analysis of Alternatives

- Payback Period (cont'd)
  - The payback period with interest is the length of time required to recover the first cost of an investment from the cash flow produced by the investment for a given interest rate i.
  - It can be computer as the smallest n that produces:

$$\sum_{t=0}^{n} F_t (1+i)^{-t} \geq 0 \qquad (69)$$

Economic Analysis of Alternatives

- **Example 28:** Payback Period
  - According to Table 9, the payback period for only the $1,000.00 disbursement without interest is 3 years.
  - The payback period for only the $1,800.00 disbursement without interest is 5 years.

Homework Assignment #6

Problems:
6.6
6.15
6.31
6.33
6.38