Introduction

“The reliability of an engineering system can be defined as its ability to fulfill its design purpose defined as performance requirements for some time period and environmental conditions. The theory of probability provides the fundamental bases to measure this ability.”
Introduction

- The reliability assessment methods can be based on
  1. Analytical strength-and-load performance functions, or
  2. Empirical life data.
- They can also be used to compute the reliability for a given set of conditions that are time invariant or for a time-dependent reliability.

Introduction

- The reliability of a component or system can be assessed in the form of a probability of meeting satisfactory performance requirements according to some performance functions under specific service and extreme conditions within a stated time period.
- Random variables with mean values, variances, and probability distribution functions are used to compute probabilities.
Analytical Performance-Based Reliability Assessment

- First-Order Second Moment (FOSM) Method.
- Advanced Second Moment Method
- Computer-Based Monte Carlo Simulation

\[ Z = Z(X_1, X_2, \ldots, X_n) = \text{Supply - Demand} \quad (1a) \]
\[ Z = Z(X_1, X_2, \ldots, X_n) = \text{Structural strength - Load effect} \quad (1b) \]
\[ Z = Z(X_1, X_2, \ldots, X_n) = R-L \quad (1c) \]

- \( Z \) = performance function of interest
- \( R \) = the resistance or strength or supply
- \( L \) = the load or demand as illustrated in Figure 1
Analytical Performance-Based Reliability Assessment

Advanced Second-Moment Method

- The failure surface (or the limit state) of interest can be defined as \( Z = 0 \).
- When \( Z < 0 \), the element is in the failure state, and when \( Z > 0 \) it is in the survival state.
- If the joint probability density function for the basic random variables \( x_i \)'s is \( f_{x_1,x_2,\ldots,x_n}(x_1,x_2,\ldots,x_n) \), then the failure probability \( P_f \) of the element can be given by the integral

\[
P_f = \int \cdots \int f_{x_1,x_2,\ldots,x_n}(x_1,x_2,\ldots,x_n) dx_1 dx_2 \cdots dx_n \quad (2)
\]
Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Where the integration is performed over the region in which \( Z < 0 \).
  - In general, the joint probability density function is unknown, and the integral is a formidable task.
  - For practical purposes, alternate methods of evaluating \( Pf \) are necessary. Reliability is assessed as one minus the failure probability.

- Reliability Index
  - Instead of using direct integration (Eq. 2), performance function \( Z \) in Eq. 1 can be expanded using Taylor series about the mean value of \( X_s \) and then truncated at the linear terms. Therefore, the first-order approximation for the mean and variance are as follows:

\[
\begin{align*}
\mu_Z & \approx Z(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}) \quad (3) \\
\sigma_Z^2 & \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial Z}{\partial X_i} \right) \left( \frac{\partial Z}{\partial X_j} \right) \text{Cov}(X_i, X_j) \quad (4a)
\end{align*}
\]
CHAPTER 4a. RELIABILITY ASSESSMENT

Analytical Performance-Based Reliability Assessment

Advanced Second-Moment Method

- Reliability Index (cont’d)

Where

\[ \mu = \text{mean of random variable} \]
\[ \mu_Z = \text{mean of } Z \]
\[ \sigma_Z^2 = \text{variance of } Z \]
\[ \text{Cov}(X_i, X_j) = \text{covariance of } X_i \text{ and } X_j \]
\[ \frac{\partial Z}{\partial X_i} = \text{partial derivative evaluated at the mean of random variable} \]

- Advanced Second-Moment Method

- Reliability Index (cont’d)

• For uncorrelated random variables, the variance can be expressed as

\[ \sigma_Z^2 \approx \sum_{i=1}^{n} \sigma_{X_i}^2 \left( \frac{\partial Z}{\partial X_i} \right)^2 \]  
(4b)

• The reliability index \( \beta \) can be computed from:

\[ \beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_L}{\sqrt{\mu_R^2 + \mu_L^2}} \]  
(5)

\[ P_f = 1 - \Phi(\beta) \]  
(6)

If \( z \) is assumed normally distributed.
Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions
    - For nonlinear performance functions, the Taylor series expansion of $Z$ in linearized at some point on the failure surface referred to as the **design point** or **checking point** or **the most likely failure point** rather than at the mean.
    - Assuming $X_i$ variables are uncorrelated, the following transformation to reduced or normalized coordinates can be used:
      \[
      Y_i = \frac{\bar{X}_i - \mu_{X_i}}{\sigma_{X_i}} \quad (8a)
      \]
Advanced Second-Moment Method

- Nonlinear Performance Functions (cont’d)
  - It can be shown that the reliability index $\beta$ is the shortest distance to the failure surface from the origin in the reduced $Y$-coordinate system.
  - The shortest distance is shown in Figure 3, and the reduced coordinates are

$$ Y_L = \frac{\sigma_R}{\sigma_L} Y_R + \frac{\mu_R - \mu_L}{\sigma_L} $$

(8b)

![Figure 3. Performance Function for a Linear, Two-Random Variable Case in Normalized Coordinates](image)
Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions (cont’d)
    - The concept of the shortest distance applies for a nonlinear performance function, as shown in Figure 4.
    - The reliability index $\beta$ and the design point, $(X'_1, X'_2, \ldots, X'_n)$, can be determined by solving the following system of nonlinear equations iteratively for $\beta$:

$$ Y_L = \frac{L - \mu_L}{\sigma_L} $$

$$ Y_R = \frac{R - \mu_R}{\sigma_R} $$

Figure 4. Performance Function for a Nonlinear, Two-Random Variable Case in Normalized Coordinates
Analytical Performance-Based Reliability Assessment

Advanced Second-Moment Method
- Nonlinear Performance Functions (cont’d)

\[
\alpha_i = \frac{\left(\frac{\partial Z}{\partial X_i}\right)\sigma_{X_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \sigma_{X_i}^2}}^{1/2} \tag{9}
\]

\[
X_i^* = \mu_{X_i} - \alpha_i \beta \sigma_{X_i} \tag{10}
\]

\[
Z(X_1^*, X_2^*, \ldots, X_n^*) = 0 \tag{11}
\]

Advanced Second-Moment Method
- Nonlinear Performance Functions (cont’d)
  - Where \( \alpha_i \) is the directional cosine, and the partial derivatives are evaluated at the design point.
  - Eq. 6 can be used to compute \( P_f \).
  - However, the above formulation is limited to normally distributed random variables.
  - The directional cosines are considered as measure of the importance of the corresponding random variables in determining the reliability index \( \beta \).
Analytical Performance-Based Reliability Assessment

Advanced Second-Moment Method

- Nonlinear Performance Functions (cont’d)
  - Also, partial safety factors $\gamma$ that are used in load and resistance factor design (LRFD) can be calculated from
    $$\gamma = \frac{X^*}{\mu_X}$$  \hspace{1cm} (12)
  - Generally, partial safety factors take on values larger than 1 for loads, and less than 1 for strengths.

Advanced Second-Moment Method

- Equivalent Normal Distributions
  - If a random variable $X$ is not normally distributed, then it must be transformed to an equivalent normally distributed random variable.
  - The parameters of the equivalent normal distribution are
    $$\mu_{X_i}^N \quad \text{and} \quad \sigma_{X_i}^N$$
  - These parameters can be estimated by imposing two conditions.
Analytical Performance-Based Reliability Assessment

**Advanced Second-Moment Method**

- **Equivalent Normal Distributions (cont’d)**

  **First condition** can be expressed as

  \[
  \Phi(\frac{X_i^* - \mu_N^i}{\sigma_N^{X_i}}) = F_i(X_i^*) \quad (13a)
  \]

  **Second condition** can be expressed as

  \[
  \phi(\frac{X_i^* - \mu_N^i}{\sigma_N^{X_i}}) = f_i(X_i^*) \quad (13b)
  \]

Analytical Performance-Based Reliability Assessment

**Advanced Second-Moment Method**

- **Equivalent Normal Distributions (cont’d)**

  where

  \[
  \begin{align*}
  F_i &= \text{non-normal cumulative distribution function} \\
  f_i &= \text{non-normal probability density function} \\
  \Phi &= \text{cumulative distribution function of the standard normal variate} \\
  \phi &= \text{probability density function of the standard normal variate.}
  \end{align*}
  \]
Analytical Performance-Based Reliability Assessment

- **Advanced Second-Moment Method**
  - Equivalent Normal Distributions (cont’d)
  - The standard deviation and mean of equivalent normal distributions are given by

\[
\sigma_{X_i}^N = \frac{\phi(\Phi^{-1}[F_i(X_i^*)])}{f_i(X_i^*)} \quad (14a)
\]

\[
\mu_{X_i}^N = X_i^* - \Phi^{-1}[F_i(X_i^*)]\sigma_{X_i}^N \quad (14b)
\]

- Once \( \sigma_{X_i}^N \) and \( \mu_{X_i}^N \) are determined for each random variable, \( \beta \) can be solved following the same procedure of Eqs. 9 through 11.
- The advanced second moment (ASM) method can deal with
  - Nonlinear performance function, and
  - Non-normal probability distributions
Advanced Second-Moment Method

- Correlated Random Variables

A correlated (and normal) pair of random variables $X_1$ and $X_2$ with a correlation coefficient $\rho$ can be transformed into noncorrelated pair $Y_1$ and $Y_2$ by solving for two eigenvalues and the corresponding eigenvectors as follows:

\[
Y_1 = \frac{1}{2t} \left( \frac{X_1 - \mu_{X_1}}{\sigma_{X_1}} + \frac{X_2 - \mu_{X_2}}{\sigma_{X_2}} \right)
\]  

(15a)

\[
Y_2 = \frac{1}{2t} \left( \frac{X_1 - \mu_{X_1}}{\sigma_{X_1}} - \frac{X_2 - \mu_{X_2}}{\sigma_{X_2}} \right)
\]  

(15b)

where $t = \sqrt{0.5}$. The resulting $Y$ variables are noncorrelated with respective variances that are equal to the eigenvalues ($\lambda$) as follows:

\[
\sigma_{Y_1}^2 = \lambda_1 = 1 + \rho
\]  

(16a)

\[
\sigma_{Y_2}^2 = \lambda_2 = 1 - \rho
\]  

(16b)
Analytical Performance-Based Reliability Assessment

- For a correlated pair of random variables, Eqs. 9 and 10, have to be revised, respectively, to

\[ \alpha_i = \frac{\left[ \frac{\partial Z}{\partial X_1} \sigma X_1 + \frac{\partial Z}{\partial X_2} t \sigma X_2 \right] \sqrt{1 + \rho}}{\left[ \left( \frac{\partial Z}{\partial X_1} \right)^2 \sigma X_1^2 + \left( \frac{\partial Z}{\partial X_2} \right)^2 \sigma X_2^2 + 2 \rho \left( \frac{\partial Z}{\partial X_1} \right) \left( \frac{\partial Z}{\partial X_2} \right) \sigma X_1 \sigma X_2 \right]^{1/2}} \] (17a)

\[ \alpha_i = \frac{\left[ \frac{\partial Z}{\partial X_1} \sigma X_1 - \left( \frac{\partial Z}{\partial X_2} \right) t \sigma X_2 \right] \sqrt{1 - \rho}}{\left[ \left( \frac{\partial Z}{\partial X_1} \right)^2 \sigma X_1^2 + \left( \frac{\partial Z}{\partial X_2} \right)^2 \sigma X_2^2 + 2 \rho \left( \frac{\partial Z}{\partial X_1} \right) \left( \frac{\partial Z}{\partial X_2} \right) \sigma X_1 \sigma X_2 \right]^{1/2}} \] (17b)

and

\[ X_1^* = \mu X_1 - \sigma X_1 \ t \beta \left( \alpha_{Y_1} \sqrt{\lambda_1} + \alpha_{Y_2} \sqrt{\lambda_2} \right) \] (18a)

\[ X_2^* = \mu X_2 - \sigma X_2 \ t \beta \left( \alpha_{Y_1} \sqrt{\lambda_1} - \alpha_{Y_2} \sqrt{\lambda_2} \right) \] (18b)

where the partial derivatives are evaluated at the design point.
Analysed Performance-Based Reliability Assessment

Advanced Second-Moment Method

- Numerical Algorithms

- The advanced second moment (ASM) method can be used to assess the reliability of a structure according to nonlinear performance function that may include non-normal random variables.

- Implementation of the method require efficient and accurate numerical algorithms.

- The ASM algorithms are provided in the following two flowcharts for
  - Noncorrelated random variables (Case a)
  - Correlated random variables (Case b)
Case b: Correlated Random Variables

Assign the mean value for each random variable as a starting point value: 
\[
(x_1, x_2, \ldots, x_n) = (\mu_1, \mu_2, \ldots, \mu_n)
\]

Compute the standard deviation and mean of the equivalent normal distribution for each non-normal random variable using Eqs. 13 and 14.

Compute the partial derivative \( \frac{dZ}{dx_i} \) for each RV using Eq. 9.

Compute the directional cosine \( \alpha_i \) for each random variable as given in Eq. 9 at the design point. For correlated pairs of random variables Eq. 17 should be used.

Compute the reliability index \( \beta \): substitute Eq. 10 (for noncorrelated) and Eq. 18 (for correlated) into Eq. 11.

Satisfy the limit state \( Z = 0 \) in Eq. 11 using a numerical root-finding method.

If \( \beta \) converges? Yes

\[ \beta \]

Take \( \beta \) value

End

**Example 1: Reliability Assessment Using a Nonlinear Performance Function**

The strength-load performance function for a component is assumed to have the following form:

\[
Z = X_1 X_2 - \sqrt{X_3}
\]

where \( X \)'s are basic random variables with the following probabilistic characteristics:

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Mean Value (( \mu ))</th>
<th>Standard Deviation (( \sigma ))</th>
<th>Coefficient of Variation</th>
<th>Case (a) Distribution Type</th>
<th>Case (b) Distribution Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>Normal</td>
<td>Lognormal</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>5</td>
<td>0.25</td>
<td>0.05</td>
<td>Normal</td>
<td>Lognormal</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>4</td>
<td>0.80</td>
<td>0.20</td>
<td>Normal</td>
<td>Lognormal</td>
</tr>
</tbody>
</table>
Analytical Performance-Based Reliability Assessment

**Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function**

- Using first-order reliability analysis based on first-order Taylor series, the following can be obtained from Eqs. 3 to 5:

\[ \mu_z = (1)(5) - \sqrt{4} = 5 - 2 = 3 \]

\[ \sigma_z = \sqrt{5^2(0.25)^2 + (1)^2(0.25)^2 + (-0.5/\sqrt{4})^2(0.8)^2} \]

\[ = \sqrt{1.5625 + 0.0625 + 0.04} = 1.2903 \]

\[ \beta = \frac{\mu_z}{\sigma_z} = \frac{3}{1.2903} = 2.325 \]

---

**Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function**

- These values are applicable to both cases (a) and (b). Using advanced second-moment reliability analysis, the following table can be constructed for cases (a) and (b):

<table>
<thead>
<tr>
<th>Case (a): Iteration 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random Variable</td>
</tr>
<tr>
<td>$X_1$</td>
</tr>
<tr>
<td>$X_2$</td>
</tr>
<tr>
<td>$X_3$</td>
</tr>
</tbody>
</table>
Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

- The derivatives in the above table are evaluated at the failure point. The failure point in the first iteration is assumed to be the mean values of the random variables.
- The reliability index can be determined by solving for the root according to Eq. 11 for the limit state of this example using the following equation:

\[ Z = \left( \mu_{X_1} - \alpha_1 \beta \sigma_{X_1} \right) \left( \mu_{X_2} - \alpha_2 \beta \sigma_{X_2} \right) - \sqrt{\mu_{X_3} - \alpha_3 \beta \sigma_{X_3}} = 0 \]

Therefore, \( \beta = 2.37735 \) for this iteration.

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>( \frac{\partial Z}{\partial X_i} \sigma_{X_i} )</th>
<th>Directional Cosines (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>4.242E-01</td>
<td>1.221E+00</td>
<td>9.841E-01</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>4.885E+00</td>
<td>1.061E-01</td>
<td>8.547E-02</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>4.295E+00</td>
<td>-1.930E-01</td>
<td>-1.555E-01</td>
</tr>
</tbody>
</table>
Analytical Performance-Based Reliability Assessment

Example 1 (cont’d): Reliability Assessment Using a Nonlinear Performance Function

- Therefore, $\beta = 2.3628$ for this iteration.

### Case (a): Iteration 3

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>$\frac{\partial Z}{\partial X_i}$</th>
<th>$\sigma_{x_i}$</th>
<th>Directional Cosines ($\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>4.187E-01</td>
<td>1.237E+00</td>
<td></td>
<td>9.846E-01</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.950E+00</td>
<td>1.047E-01</td>
<td></td>
<td>8.329E-02</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.294E+00</td>
<td>-1.930E-01</td>
<td></td>
<td>-1.536E-01</td>
</tr>
</tbody>
</table>

Analytical Performance-Based Reliability Assessment

Example 1 (cont’d): Reliability Assessment Using a Nonlinear Performance Function

- Therefore, $\beta = 2.3628$ for this iteration which means that $\beta$ has converged to 2.3628.
- The failure probability $=1-\Phi(\beta) = 0.009068$.
- The partial safety factors can be computed as:

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Partial Safety Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0.418378</td>
<td>0.418378</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.950849</td>
<td>0.99017</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.290389</td>
<td>1.072597</td>
</tr>
</tbody>
</table>
Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

- Case (b)
  - The parameters of the lognormal distribution can be computed for three random variables based on their respective means (µ) and deviations (σ) as follows:

\[
\sigma_Y^2 = \ln \left[ 1 + \left( \frac{\sigma_X}{\mu_X} \right)^2 \right] \quad \text{and} \quad \mu_Y = \ln(\mu_X) - \frac{1}{2} \sigma_Y^2
\]

Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

- The results of these computations are summarized as follows:

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Distribution Type</th>
<th>First Parameter (µ_Y)</th>
<th>Second Parameter (σ_Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_1</td>
<td>Lognormal</td>
<td>-0.03031231</td>
<td>0.24622068</td>
</tr>
<tr>
<td>X_2</td>
<td>Lognormal</td>
<td>1.608189472</td>
<td>0.04996879</td>
</tr>
<tr>
<td>X_3</td>
<td>Lognormal</td>
<td>1.366684005</td>
<td>0.20</td>
</tr>
</tbody>
</table>
Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

Case (b): Iteration 1

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Standard Deviation</th>
<th>Mean Value</th>
<th>$\frac{\partial Z}{\partial X_i} \sigma_N$</th>
<th>Directional Cosines ($\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>1.000E+00</td>
<td>2.462E-01</td>
<td>9.697E-01</td>
<td>1.231E+00</td>
<td>9.681E-01</td>
</tr>
<tr>
<td>$X_2$</td>
<td>5.000E+00</td>
<td>2.498E-01</td>
<td>4.994E+00</td>
<td>2.498E-01</td>
<td>1.965E-01</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.000E+00</td>
<td>7.922E-01</td>
<td>3.922E+00</td>
<td>-1.980E-01</td>
<td>-1.557E-01</td>
</tr>
</tbody>
</table>

• The derivatives in the above table are evaluated at the failure point. The failure point in the first iteration is assumed to be the mean values of the random variables.

• The reliability index can be determined by solving for the root according to Eq. 11 for the limit state of this example using the following equation:

$$Z = \left( \mu_{X_1}^N - \alpha_1 \beta \sigma_{X_1}^N \right) \left( \mu_{X_2}^N - \alpha_2 \beta \sigma_{X_2}^N \right) - \sqrt{\mu_{X_3}^N - \alpha_3 \beta \sigma_{X_3}^N} = 0 $$
Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

- Therefore, $\beta = 2.30530$ for this iteration.

### Case (b): Iteration 2

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Standard Deviation</th>
<th>Mean Value</th>
<th>$\frac{\partial Z}{\partial X_i} \sigma_{X_i}$</th>
<th>Directional Cosines ($\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>4.202E-01</td>
<td>1.035E-01</td>
<td>7.718E-01</td>
<td>5.050E-01</td>
<td>9.118E-01</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.881E+00</td>
<td>2.439E-01</td>
<td>4.992E+00</td>
<td>1.025E-01</td>
<td>1.850E-01</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.206E+00</td>
<td>8.330E-01</td>
<td>3.912E+00</td>
<td>-2.031E-01</td>
<td>-3.667E-01</td>
</tr>
</tbody>
</table>

Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

- Therefore, $\beta = 3.3224$ for this iteration.

### Case (b): Iteration 3

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Standard Deviation</th>
<th>Mean Value</th>
<th>$\frac{\partial Z}{\partial X_i} \sigma_{X_i}$</th>
<th>Directional Cosines ($\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>4.584E-01</td>
<td>1.129E-01</td>
<td>8.020E-01</td>
<td>5.465E-01</td>
<td>9.118E-01</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.843E+00</td>
<td>2.420E-01</td>
<td>4.991E+00</td>
<td>1.109E-01</td>
<td>1.850E-01</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.927E+00</td>
<td>9.758E-01</td>
<td>3.803E+00</td>
<td>-2.198E-01</td>
<td>-3.667E-01</td>
</tr>
</tbody>
</table>
Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

Therefore, $\beta = 3.3126$ for this iteration.

Case (b): Iteration 4

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Standard Deviation</th>
<th>Mean Value</th>
<th>$\frac{\partial Z}{\partial X_i} \sigma_{X_i}$</th>
<th>Directional Cosines ($\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>4.612E-01</td>
<td>1.136E-01</td>
<td>8.041E-01</td>
<td>5.499E-01</td>
<td>9.118E-01</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.843E+00</td>
<td>2.420E-01</td>
<td>4.991E+00</td>
<td>1.116E-01</td>
<td>1.850E-01</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.989E+00</td>
<td>9.880E-01</td>
<td>3.789E+00</td>
<td>-2.212E-01</td>
<td>-3.667E-01</td>
</tr>
</tbody>
</table>

Analytical Performance-Based Reliability Assessment

Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

Therefore, $\beta = 3.3125$ for this iteration.

Case (b): Iteration 5

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Standard Deviation</th>
<th>Mean Value</th>
<th>$\frac{\partial Z}{\partial X_i} \sigma_{X_i}$</th>
<th>Directional Cosines ($\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>4.612E-01</td>
<td>1.136E-01</td>
<td>8.041E-01</td>
<td>5.500E-01</td>
<td>9.118E-01</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.843E+00</td>
<td>2.420E-01</td>
<td>4.991E+00</td>
<td>1.116E-01</td>
<td>1.850E-01</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.989E+00</td>
<td>9.880E-01</td>
<td>3.789E+00</td>
<td>-2.212E-01</td>
<td>-3.667E-01</td>
</tr>
</tbody>
</table>
Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

- Therefore, $\beta = 3.3125$ for this iteration which means that $\beta$ has converged to 3.3125.
- The failure probability $= 1 - \Phi(\beta) = 0.0004619$.
- The partial safety factors can be computed as:

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Failure Point</th>
<th>Partial Safety Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0.461189</td>
<td>0.461189</td>
</tr>
<tr>
<td>$X_2$</td>
<td>4.843135</td>
<td>0.968627</td>
</tr>
<tr>
<td>$X_3$</td>
<td>4.988968</td>
<td>1.247242</td>
</tr>
</tbody>
</table>

Monte Carlo Simulation Methods

- Monte Carlo simulation (MCS) techniques are basically sampling processes that are used to estimate the failure probability of a component or system.
- The basic random variables in Eq. 1, that is

$$Z = Z(X_1, X_2, \ldots, X_n) = R-L$$

are randomly generated and substituted into above equation.
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Then the fraction of the cases that resulted in failure are determined to assess the failure probability.
  - Three methods are described herein:
    1. Direct Monte Carlo Simulation
    2. Conditional Expectation
    3. The Importance Sampling Reduction Method

Direct Monte Carlo Simulation Method
- In this method, samples of the basic noncorrelated variables are drawn according to their corresponding probabilities characteristics and fed into performance function $Z$ as given by Eq. 1.
- Assuming that $N_f$ is the number of simulation cycles for which $Z < 0$ in $N$ simulation cycles, then an estimate of the mean failure probability can be expressed as

$$P_f = \frac{N_f}{N} \quad (19)$$
Monte Carlo Simulation Methods (cont’d)

- The variance of the estimated failure probability can be approximately computed using the variance expression for a binomial distribution as:

\[
Var(\bar{P}_f) = \frac{(1 - \bar{P}_f)\bar{P}_f}{N} \quad (20)
\]

- Therefore, the coefficient of variation (COV) of the estimated failure probability is

\[
COV(\bar{P}_f) = \frac{1}{\bar{P}_f} \sqrt{\frac{(1 - \bar{P}_f)\bar{P}_f}{N}} \quad (21)
\]

- Some of the advantages of this method is that it is easy to implement and understand.

- The disadvantages include:
  - Expensive in some cases, especially if the failure probabilities are small.
  - Inefficient

- The importance sampling method (IS) is described later for the purpose of increasing the efficiency of the IS method.
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Conditional Expectation
    - This method can also be used to estimate the failure probability according to the performance function of Eq. 1.
    - The method requires generating all the basic random variables in Eq. 1 except the random variables with the highest variability (i.e., COV), which is used as a control variable, $X_k$.
    - The conditional expectation is computed as the cumulative distribution function.

For the following performance function:

\begin{align*}
Z &= R - L \\
F_R (l_i) &= F_{R} (l_i) \\
1 - F_L (r_i) &= P_{f_i} 
\end{align*}

and for a randomly generated value of $L$ or $R$, the failure probability for each cycle is given, respectively, as

\begin{align*}
P_{f_i} &= F_R (l_i) \\
1 - F_L (r_i) &= P_{f_i}
\end{align*}
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Conditional Expectation (cont’d)
    - In these equations, \( L \) and \( R \) are the control variables. The total failure probability \( P_f \) can be estimated from
      \[
      \bar{P}_f = \frac{\sum_{i=1}^{N} P_{f_i}}{N}
      \]  
      (25)
    - Where \( N \) is the number of simulation cycles.

- The accuracy of Eq. 25 can be estimated through the variance and coefficient of variation as given by

  \[
  \text{Var}(\bar{P}_f) = \frac{\sum_{i=1}^{N} (P_{f_i} - \bar{P}_f)^2}{N(N-1)}
  \]  
  (26)

  \[
  \text{COV}(\bar{P}_f) = \frac{\sqrt{\text{Var}(\bar{P}_f)}}{\bar{P}_f}
  \]  
  (27)
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Importance Sampling (cont’d)
    • To improve the efficiency of simulation when estimating the probability of failure for a given performance function, Importance Sampling (IS) techniques are used.
    • In some performance function, if the margin of safety $Z$ is large and its variance is too small, larger simulation effort will be required to obtain sufficient simulation runs with satisfactory performances, i.e.,

  smaller failure probabilities require larger number of simulation cycles

- Importance density function, $h_x(x)$

  With mean values that are closer to the design point than their original (actual) probability distributions.
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Importance Sampling (cont’d)
    - The fundamental equation for this method is given by
      \[
      \overline{P_f} = \frac{1}{N} \sum_{i=1}^{N} I_i \frac{f_X(x_{1i}, x_{2i}, \ldots, x_{ni})}{h_X(x_{1i}, x_{2i}, \ldots, x_{ni})}
      \]  
      (28)

  - Where:
    - \( N \) = number of simulation cycles
    - \( f_X(x_{1i}, x_{2i}, \ldots, x_{ni}) \) = original joint density function of the basic random variables evaluated at the \( i \)th generated values of the basic random variables
    - \( h_X(x_{1i}, x_{2i}, \ldots, x_{ni}) \) = selected joint density function of the basic random variables evaluated at the \( i \)th generated values of the basic random variables
    - \( I_i \) = performance indicator function that takes values of either 0 for failure and 1 for survival

- The coefficient of variation of the estimate failure probability can be based on the variance of a sample mean as follows:

  \[
  COV(\overline{P_f}) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} \left( I_i \frac{f_X(x_{1i}, x_{2i}, \ldots, x_{ni})}{h_X(x_{1i}, x_{2i}, \ldots, x_{ni})} - \overline{P_f} \right)^2} \]  
  (29)

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Correlated Random Variables
    - A correlated (and normal) pair of random variables $X_1$ and $X_2$ with a correlation coefficient $\rho$ can be transformed using linear regression transformation as follows:
    \[
    X_2 = b_0 + b_1 X_1 + \varepsilon \tag{30a}
    \]
    
    - $b_0$ = intercept of a regression line between $X_1$ and $X_2$
    - $b_1$ = slope of the regression line
    - $\varepsilon$ = random (standard) error with a mean of zero and a standard deviation as given in Eq. 30d).

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Correlated Random Variables (cont’d)
    - These regression model parameters can be determined in terms of the probabilistic characteristics of $X_1$ and $X_2$ as follows:
      \[
      b_1 = \frac{\rho \sigma_{X_2}}{\sigma_{X_1}} \tag{30b}
      \]
      \[
      b_0 = \mu_{X_2} - b_1 \mu_{X_1} \tag{30b}
      \]
      \[
      \sigma_{\varepsilon} = \sigma_{X_2} \sqrt{1 - \rho^2} \tag{30d}
      \]
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Correlated Random Variables (cont’d)

**Procedure for a correlated pair of random variables:**

1. Compute the intercept of a regression line between $X_1$ and $X_2$ ($b_0$), the slope of the regression line ($b_1$), and the standard deviation of the random (standard) error ($\varepsilon$) using Eqs. 30b to 30d.
2. Generate a random (standard) error using a zero mean and a standard deviation as given by Eq. 30d.
3. Generate a random value for $X_1$ using its probabilistic characteristics (mean and variance).
4. Compute the corresponding value of $X_2$ as follows (based on Eq. 30a): $x_2 = b_0 + b_1 x_1 + \varepsilon$

   where $b_0$ and $b_1$ are computed in step 1; $\varepsilon$ is a generated random (standard) error from step 2; and $x_1$ is generated value from step 3.
5. Use the resulting random (but correlated) values of $x_1$ and $x_2$ in the simulation-based reliability assessment methods.
Analytical Performance-Based Reliability Assessment

Monte Carlo Simulation Methods

- Time-Dependent Reliability Analysis
  - Several methods for analytical time-dependent reliability assessment are available.
  - In these methods, significant structural loads as a sequence of pulses that can be described by a Poisson process with mean occurrence rate, $\lambda$, random intensity, $S$, and duration, $\tau$.
  - The limit state of the structure at any time can be defined as
    \[ R(t) - S(t) < 0 \quad (31) \]

 where $R(t)$ is the strength of the structure at time $t$ and $S(t)$ is the loads at time $t$.

The instantaneous probability of failure can then be defined at time $t$ as probability of $R(t)$ less than $S(t)$.

The reliability function, $L(t)$, was defined as the probability that the structure survives during interval of time $(0,t)$ as

\[ L(t) = \int_0^\infty \exp\left[-\lambda t \left[1 - \frac{1}{t} \int_0^t F_s(g(t)r)dr\right]\right] f_R(r)dr \quad (32a) \]
Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Time-Dependent Reliability Analysis (cont’d)
    - where \( f_R(r) \) is the probability density function of an initial strength, \( R \), and \( g(t) \) is the time-dependent degradation in strength.
    - The reliability can be expressed in terms of the conditional failure rate or hazard function, \( h(t) \) as
      \[
      h(t) = -\frac{d}{dt} \ln L(t)
      \] (32b)
      or
      \[
      L(t) = \exp\left[-\int_0^t h(\xi)d\xi\right]
      \] (32c)

- The reliability \( L(t) \) is based on the complete survival during the service life interval \((0,t)\).
- It means the probability of successful performance during a service life interval \((0,t)\).
- Therefore, the probability of failure, \( P_f(t) \), can be computed as the probability of the complementary event, i.e., \( P_f(t) = 1 - L(t) \) being not equivalent to \( P[R(t) < S(t)] \).
Empirical Reliability Analysis
Using Life Data

- **Failure and Repair**
  - The basic notion of reliability analysis based on life data is *time to failure*.
  - The useful life of a product can be measured in terms of its time to failure.
  - In addition to time, other possible exposure measures include the number of cycles to failure of mechanical, electrical, temperature or humidity.

- **Failure and Repair (cont’d)**
  - If the failed product is subject to repair or replacement, it is called *repairable* (in opposite to *non-repairable* objects).
  - The respective repair or replacement requires some time to get done, which is called *time to repair/replace*.
  - The time to failure is used for the non-repairable components or systems.
Empirical Reliability Analysis
Using Life Data

■ Failure and Repair (cont’d)
  – For repairable products, there is another important characteristic, which is called time between failures.
  – This is another random variable or a set of random variables.
  – It can be assumed that the time to the first failure is the same random variable as the time between the first and the second failures, the time between the second and the third failures, and so on.

■ Types of Data
  – Failure data often contain not only times to failure (the so-called distinct failures), but also times in use (or exposure length of time) that do not terminate with failures.
  – Such exposure time intervals terminating with non-failure are called times to censoring (TTC).
  – Therefore, life data of equipment can be classified into two types, complete and censored data
Empirical Reliability Analysis Using Life Data

- Types of Data (cont’d)
  - Failure data often contain not only times to failure (the so-called distinct failures), but also times in use (or exposure length of time) that do not terminate with failures.
  - Such exposure time intervals terminating with non-failure are called *times to censoring* (TTC).
  - Therefore, life data of equipment can be classified into two types, *complete* and *censored* data.

- The *complete life data* are commonly based on equipment tested to failure or times to failure based on equipment use, i.e., field data.
  - *Censored life data* include some observation results that represent only lower or upper limits on observation of times to failure.
Empirical Reliability Analysis
Using Life Data

Types of Data (cont’d)

– Censored data can be further classified into
  • Type I or
  • Type II

– **Type I** data are based on observations of a life test, which for economical or other reasons, must be terminated at specified time $t_0$.

– As the result, only the lifetimes of those units that have failed before $t_0$ are known exactly.

– If, during the time interval $(0, t_0]$, $s$ out of $n$ sample units failed, then the information in the data set obtained consists of $s$ observed, ordered times to failure as follows:

$$t_1 < t_2 < \ldots < t_s$$  \hspace{1cm} (33a)

– and the information that $(n - s)$ units have survived the time $t_0$. 

Empirical Reliability Analysis
Using Life Data

- Types of Data (cont’d)
  - In some life data testing, testing is continued until a specified number of failures $r$ is achieved, i.e., the respective test or observation is terminated at the $r^{th}$ failure.
  - In this case, $r$ is not random.
  - This type of testing, i.e., observation or field data collection, results in **Type II** censoring.

- Types of Data (cont’d)
  - It includes $r$ observed ordered times to failure
    \[ t_1 < t_2 < \ldots < t_r \]  
    (33b)
  - And the information that $(n - r)$ units have survived the time $t_r$.
  - But, in opposite to Type I censoring, the test or observation duration $t_r$ is random, which should be taken into account in the respective statistical estimation procedures.
Empirical Reliability Analysis
Using Life Data

- Types of Data (cont’d)
  - In reliability engineering, Type I right-censored data are commonly encountered.
  - Figure 5 shows a summary of these data types.
  - Other types of data are possible such as random censoring.

Figure 5. Types of Life Data
**Empirical Reliability Analysis Using Life Data**

**Example 2: Data of Distinct Failures**

- In this example, the following complete sample of 19 times to failure for a structural component given in years to failure is provided for illustration purposes:

26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 43, 50, 56

**Example 3: Right Censored Data**

- In this example, tests of equipment are used for demonstration purposes to produce observations in the form of life data as given in Table 1.

- The data in the table provide an example of Type I censored data (the sample size is 12), with time to censoring equal to 51 years.

- If the data collection was assumed to terminate just after the 8th failure, the data would represent a sample of Type II right censored data.
Empirical Reliability Analysis
Using Life Data

- Example 3 (cont'd): Right Censored Data
censored data with the same sample size of 12.
- The respective data are given in Table 2.

Table 1. Example of Type I Right Censored Data (in Years) for Equipment

<table>
<thead>
<tr>
<th>Time Order Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (Years)</td>
<td></td>
<td>14</td>
<td>15</td>
<td>18</td>
<td>31</td>
<td>37</td>
<td>40</td>
<td>46</td>
<td>51</td>
<td>51</td>
<td>51</td>
<td>51</td>
</tr>
<tr>
<td>TTF or TTC</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTC</td>
<td>TTC</td>
<td>TTC</td>
<td>TTC</td>
<td></td>
</tr>
</tbody>
</table>

TTF = time to failure, and TTC = time to censoring

---

Empirical Reliability Analysis
Using Life Data

- Example 3 (cont'd): Right Censored Data

Table 2. Example of Type II Right Censored Data (in Years) for Equipment

<table>
<thead>
<tr>
<th>Time Order Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (Years)</td>
<td>7</td>
<td>14</td>
<td>15</td>
<td>18</td>
<td>31</td>
<td>37</td>
<td>40</td>
<td>46</td>
<td>46</td>
<td>46</td>
<td>46</td>
<td>46</td>
</tr>
<tr>
<td>TTF or TTC</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTF</td>
<td>TTC</td>
<td>TTC</td>
<td>TTC</td>
<td>TTC</td>
<td></td>
</tr>
</tbody>
</table>

TTF = time to failure, and TTC = time to censoring
Empirical Reliability Analysis
Using Life Data

■ Example 4: Random Censoring

- Table 3 contains the time to failure data, in which two failure modes were observed.
- The data in this example were generated using Monte Carlo simulation.
- The simulation process is restarted once a failure occurs according to one of the modes at time $t$, making this time $t$ for the other mode as a time to censoring.

<table>
<thead>
<tr>
<th>Year</th>
<th>TTF (Years)</th>
<th>Number of Occurrences of a Given Failure Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Strength (FM1)</td>
</tr>
<tr>
<td>1984</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1985</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>1986</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>1987</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>1988</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>1989</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>1990</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>1991</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>1992</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>1993</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>1994</td>
<td>11</td>
<td>5</td>
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<td>1995</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>1996</td>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>1997</td>
<td>14</td>
<td>2</td>
</tr>
<tr>
<td>1998</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>1999</td>
<td>16</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3.
Partial Data Set From 20,000 Simulation Cycles for the Two Failure Modes of Strength and Fatigue for a Structural Component