CHAPTER

5a

REINFORCED CONCRETE

A Fundamental Approach - Fifth Edition

FLEXURE IN BEAMS

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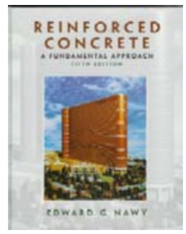

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ENCE 454 – Design of Concrete Structures

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Introduction

- Bending moment produces bending strains on a beam, and consequently compressive and tensile stresses.
- Under positive moment (as normally the case), compressive stresses are produced in the top of the beam and tensile stresses are produced in the bottom.
- Bending members must resist both compressive and tensile stresses.

Introduction (cont'd)

- Stresses in a Beam

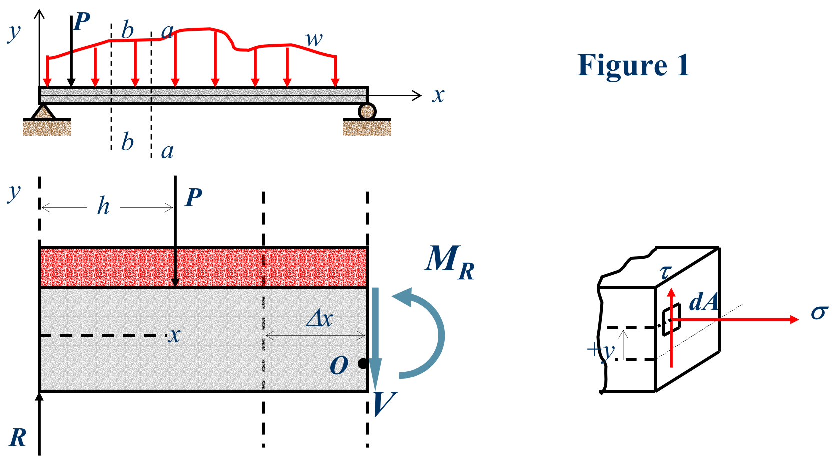
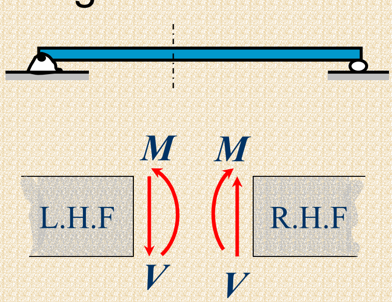


Figure 1

Introduction (cont'd)

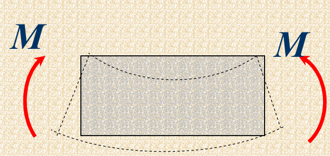
- Sign Convention

Figure 2



(a) Positive Shear & Moment

(b) Positive Shear (clockwise)

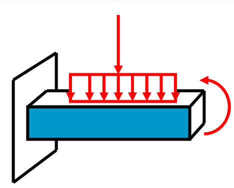


(c) Positive Moment (concave upward)

Beams: Mechanics of Bending Review

- Introduction
  - The most common type of structural member is a beam.
  - In actual structures beams can be found in an infinite variety of
    - Sizes
    - Shapes, and
    - Orientations

Beams: Mechanics of Bending Review

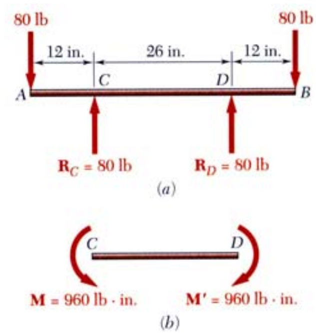


- Introduction

  Definition

A beam may be defined as a member whose length is relatively large in comparison with its thickness and depth, and which is loaded with transverse loads that produce significant bending effects as oppose to twisting or axial effects

Beams: Mechanics of Bending Review



Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Beams: Mechanics of Bending Review

- Flexural Normal Stress

For flexural loading and linearly elastic action, the neutral axis passes through the centroid of the cross section of the beam.

Beams: Mechanics of Bending Review

- The elastic flexural formula for normal stress is given by

$$f_b = \frac{Mc}{I} \qquad (1)$$

where

f_b = calculated bending stress at outer fiber of the cross section

M = the applied moment

c = distance from the neutral axis to the outside tension or compression fiber of the beam

I = moment of inertia of the cross section about neutral axis

Beams: Mechanics of Bending Review

- By rearranging the flexure formula, the maximum moment that may be applied to the beam cross section, called the <u>*resisting moment, M_R*</u>, is given by

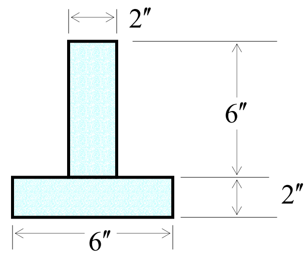
$$M_R = \frac{F_b I}{c} \qquad (2)$$

Where F_b = the allowable bending stress

Beams: Mechanics of Bending Review

- Example 1

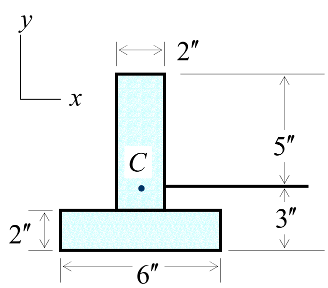
  Determine the maximum flexural stress produced by a resisting moment M_R of +5000 ft-lb if the beam has the cross section shown in the figure.



Beams: Mechanics of Bending Review

- Example 1 (cont'd)

  First, we need to locate the neutral axis from the bottom edge:



$$y_C = \frac{(1)(2\times 6)+(2+3)(2\times 6)}{2\times 6+2\times 6} = \frac{72}{24} = 3''$$

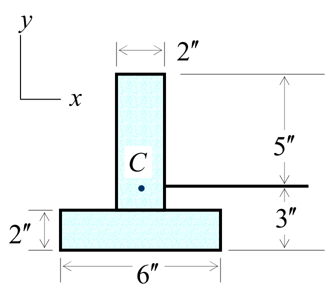
$$y_{\text{ten}} = 3'' \quad y_{\text{com}} = 6+2-3 = 5'' = y_{\max} = c$$

$$\text{Max. Stress} = f_b = \frac{Mc}{I}$$

Beams: Mechanics of Bending Review

- Example 1 (cont'd)

  Find the moment of inertia I with respect to the x axis using parallel axis-theorem:



$$I = \frac{6(2)^3}{12} + (6 \times 2)(2)^2 + \frac{2(6)^3}{12} + (2 \times 6)(3-1)^2$$

$$= 4 + 48 + 36 + 48 = 136 \text{ in}^4$$

$$\text{Max. Stress (com)} = \frac{(5 \times 12)(5)}{136} = \underline{2.21 \text{ ksi}}$$

Beams: Mechanics of Bending Review

- Internal Couple Method
  - The procedure of the flexure formula is easy and straightforward for a beam of known cross section for which the moment of inertia I can be found.
  - However, for a reinforced concrete beam, the use of the flexure formula can be somewhat complicated.
  - The beam in this case is not homogeneous and concrete does not behave elastically.

Beams: Mechanics of Bending Review

- Internal Couple Method (cont'd)
  - In this method, the couple represents an internal resisting moment and is composed of a compressive force *C* and a parallel internal tensile force *T* as shown in Fig. 1.
  - These two parallel forces *C* and *T* are separated by a distance *Z*, called the moment arm. (Fig. 1)
  - Because that all forces are in equilibrium, therefore, *C* must equal *T*.

Beams: Mechanics of Bending Review

- Internal Couple Method (cont'd)

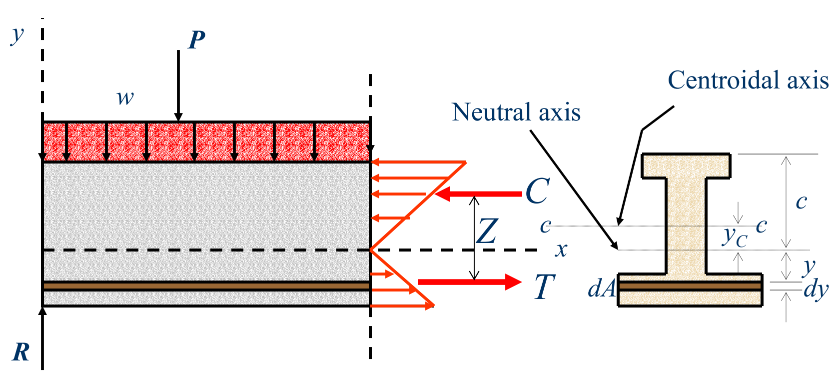


Figure 1

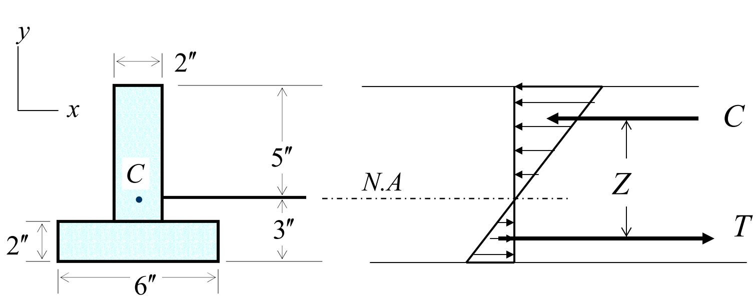
Beams: Mechanics of Bending Review

- Internal Couple Method (cont'd)
  - The internal couple method of determining beam stresses is more general than the flexure formula because it can be applied to homogeneous or non-homogeneous beams having linear or nonlinear stress distributions.
  - For reinforced concrete beam, it has the advantage of using the basic resistance pattern that is found in a beam.

Beams: Mechanics of Bending Review

- Example 2

  Repeat Example 1 using the internal couple method.



Beams: Mechanics of Bending Review

- Example 2 (cont'd)
  - Because of the irregular area for the tension zone, the tensile force T will be broken up into components T_1, T_2, and T_3.
  - Likewise, the moment arm distance Z will be broken up into components Z_1, Z_2, and Z_3, and calculated for each component tensile force to the compressive force C as shown in Fig. 2.

Beams: Mechanics of Bending Review

- Example 2 (cont'd)

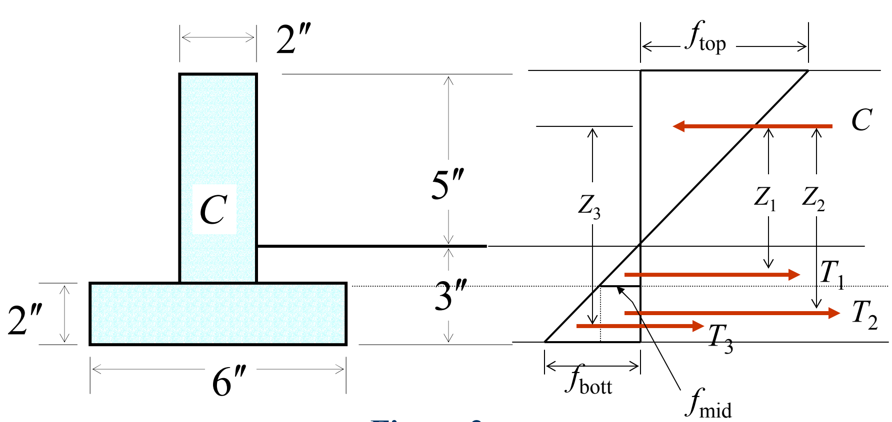
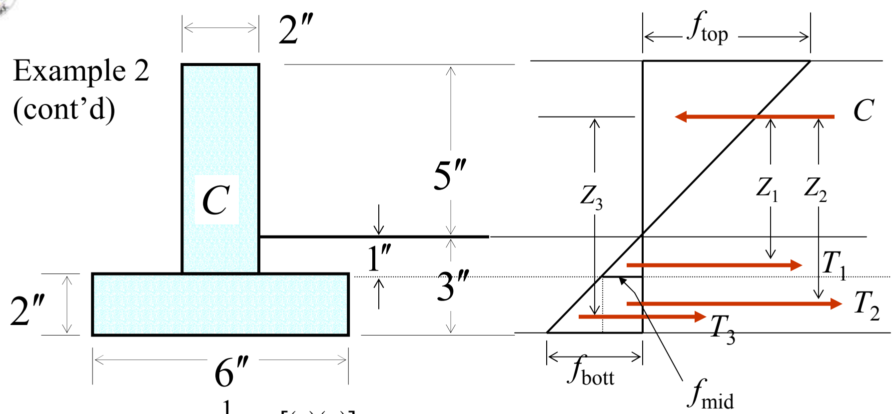


Figure 2

Example 2
(cont'd)



$$C = f_{avg} \times \text{area} = \frac{1}{2} f_{top}[(5)(2)] = 5 f_{top}$$

$$T_1 = f_{avg} \times \text{area} = \frac{1}{2} f_{mid}[(1)(2)] = f_{mid} = \frac{1}{3} f_{bott}$$

$$T_2 = f_{avg} \times \text{area} = f_{mid}[(2)(6)] = 12 f_{mid} = 4 f_{bott}$$

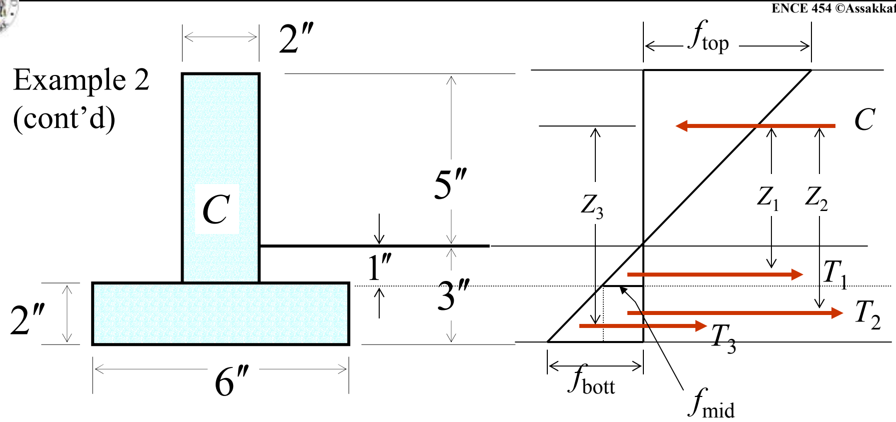
$$T_3 = f_{avg} \times \text{area} = \left(\frac{f_{bott} - f_{mid}}{2}\right)[(2)(6)] = 6 f_{bott} - 6 f_{mid}$$

From similar triangles:

$$\frac{f_{mid}}{f_{bott}} = \frac{1}{3}$$

$$\therefore f_{mid} = \frac{1}{3} f_{bott}$$

Example 2
(cont'd)



$$C = T = T_1 + T_2 + T_3$$

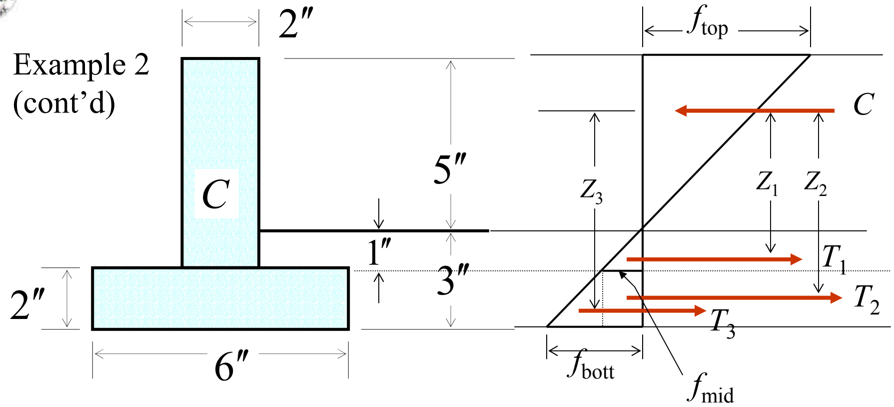
$$5 f_{top} = \frac{1}{3} f_{bott} + 4 f_{bott} + 6 f_{bott} - 6 f_{mid}$$

$$5 f_{top} = \frac{1}{3} f_{bott} + 4 f_{bott} + 6 f_{bott} - 2 f_{bott} = \frac{25}{3} f_{bott}$$

→

$$f_{top} = \frac{5}{3} f_{bott}$$

Example 2 (cont'd)



$$Z_1 = \frac{2}{3}(5) + \frac{2}{3}(1) = 4 \text{ in.}$$

$$Z_2 = \frac{2}{3}(5) + 2 = \frac{16}{3} \text{ in.}$$

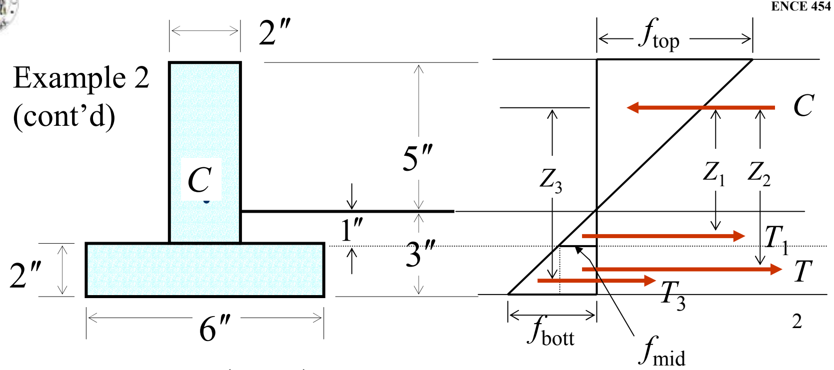
$$Z_3 = \frac{2}{3}(5) + 1 + \frac{2}{3}(2) = \frac{17}{3} \text{ in.}$$

$$M_{ext} = M_R$$

$$5000(12) = Z_1T_1 + Z_2T_2 + Z_3T_3$$

$$60{,}000 = Z_1T_1 + Z_2T_2 + Z_3T_3$$

Example 2 (cont'd)



$$60{,}000 = 4\left(\frac{1}{3} f_{bott}\right) + \frac{16}{3}(4 f_{bott}) + \frac{17}{3}(4 f_{bott}) = \frac{136}{3} f_{bott}$$

Therefore,

$$f_{bott} = 1{,}323.53 \text{ psi (Tension)}$$

The maximum Stress is compressive stress :

$$f_{max} = f_{top} = \frac{5}{3} f_{bott} = \frac{5}{3}(1{,}323.53) = 2{,}205.88 \text{ psi} = 2.21 \text{ ksi (Com)}$$

Methods of Analysis and Design

- Elastic Design
  - Elastic design is considered valid for the homogeneous plain concrete beam as long as the tensile stress does not exceed the modulus of rupture f_r.
  - Elastic design can also be applied to a reinforced concrete beam using the working stress design (WSD) or allowable stress design (ASD) approach.

Methods of Analysis and Design

- WSD or ASD Assumptions
  - A plain section before bending remains plane after bending.
  - Stress is proportional to strain (Hooke's Law).
  - Tensile stress for concrete is considered zero and reinforcing steel carries all the tension.
  - The bond between the concrete and steel is perfect, so no slip occurs.

Methods of Analysis and Design

- Strength Design Method
  - This method is the modern approach for the analysis and design of reinforced concrete.
  - The assumption are similar to those outlined for the WSD or ASD with one exception:
    - Compressive concrete stress is approximately proportional to strain up to moderate loads. As the load increases, the approximate proportionality ceases to exit, and the stress diagram takes a shape similar to the concrete stress-strain curve of the following figure.

Methods of Analysis and Design

- Concrete Compressive Strength

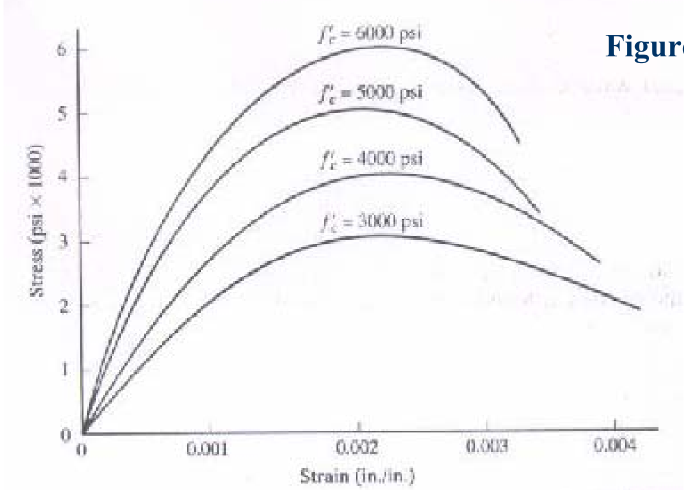


Figure 3

Methods of Analysis and Design

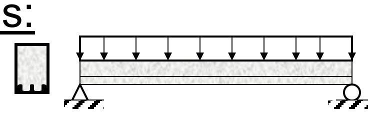
- Comparison between the Two Methods

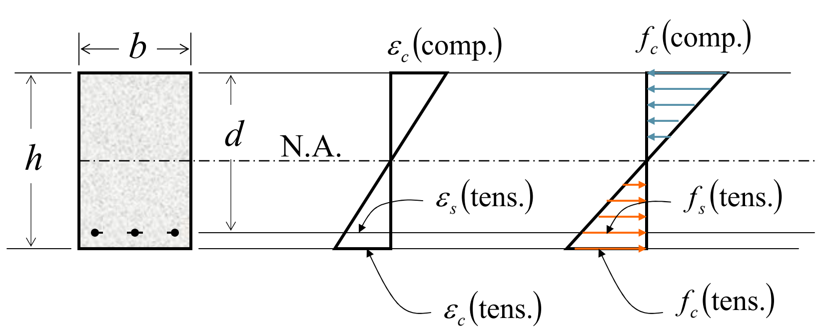
WSD or ASD	USD
▪ Working (service) loads are used and a member is designed based on an allowable compressive bending stress, normally 0.45 f'_c ▪Compressive stress pattern is assumed to vary linearly from zero at the neutral axis. ▪Formula: $$\frac{R_n}{\text{FS}} \geq \sum_{i=1}^{m} L_i$$ ASD	▪Service loads are amplified using partial safety factors. ▪A member is design so that its strength is reduced by a reduction safety factor. ▪The strength at failure is commonly called the ultimate strength ▪Formula: $$\phi R_n \geq \sum_{i=1}^{m} \gamma_i L_i$$ LRFD

Behavior Under Load

(1) At very small loads:

Stresses Elastic and Section Uncracked



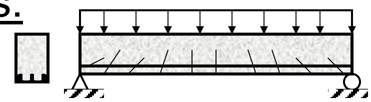


Stresses are below modulus of rupture.

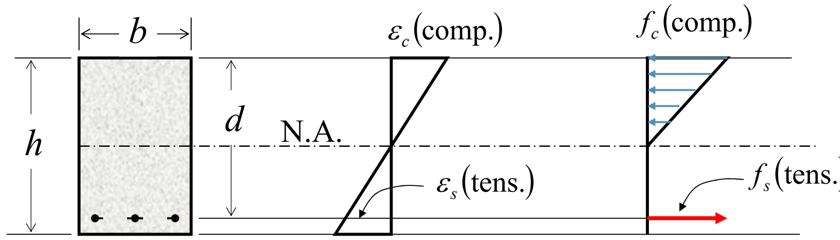
Behavior Under Load

(2) At moderate loads:

Stresses Elastic and Section Cracked



Reinforced Concrete Beam



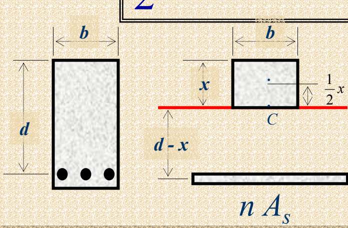
- Tensile stresses of concrete will be exceeded.
- Concrete will crack (hairline crack), and steel bars will resist tensile stresses.
- This will occur at approximately $0.5 f'_c$.

Behavior Under Load

- Reinforced Concrete Beam Formula

The neutral axis for a concrete beam is found by solving the quadratic equation:

$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0 \qquad (1)$$

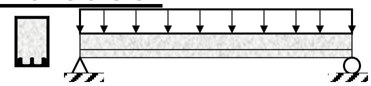


$$n = \frac{E_s}{E_c}$$

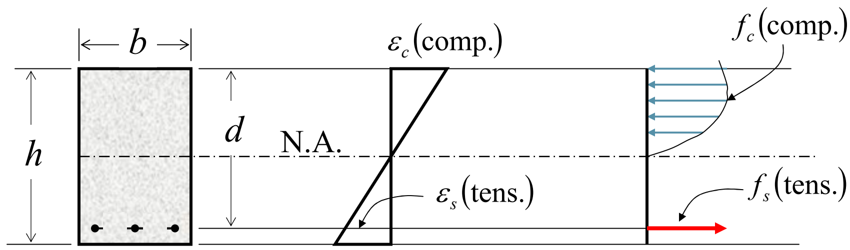
Behavior Under Load

(3) With further load increase:

Flexural Strength
ACI Approach



Reinforced Concrete Beam



- Stress curve above N.A. will be similar to the stress-strain curve of Fig. 3.
- Concrete has cracked, and the process is irreversible.
- Steel bar has yielded and will not return to its original length.

Strength Design Method

Assumptions

- Basic Assumption:
  - A plane section before bending remains plane after bending.
  - Stresses and strain are approximately proportional up to moderate loads (concrete stress $\leq 0.5\ f_c'$). When the load is increased, the variation in the concrete stress is no longer linear.
  - Tensile strength of concrete is neglected in the design of reinforced concrete beams.

Strength Design Method

Assumptions

- Basic Assumption (cont'd):
  4. The maximum usable concrete compressive strain at the extreme fiber is assumed equal to 0.003 (Fig. 4)
  5. The steel is assumed to be uniformly strained to the strain that exists at the level of the centroid of the steel. Also if the strain in the steel ε_s is less than the yield strain of the steel ε_y, the stress in the steel is $E_s\ \varepsilon_s$. If $\varepsilon_s \geq \varepsilon_y$, the stress in steel will be equal to f_y (Fig. 5)

Strength Design Method

Assumptions

- Basic Assumption (cont'd):
  6. The bond between the steel and concrete is perfect and no lip occurs.

Figure 4

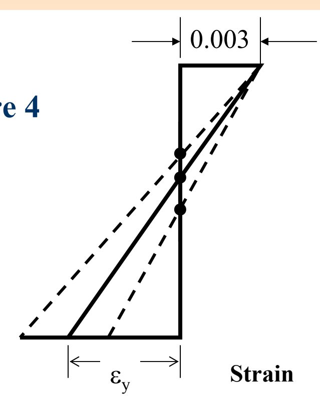
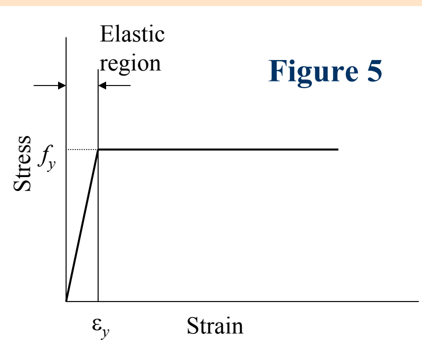


Figure 5



Idealized Stress-Strain Curve

Flexural Strength of Rectangular Beams

- Ultimate Moment Strength Capacity
  - The ultimate moment for a reinforced concrete beam can be defined as the moment that exists just prior to the failure of the beam.
  - In order to evaluate this moment, we have to examine the strains, stresses, and forces that exist in the beam.
  - The beam of Fig. 6 has a width of b, an effective depth d, and is reinforced with a steel area A_s.

Flexural Strength of Rectangular Beams

- Ultimate Strength

Flexural Strength ACI Approach

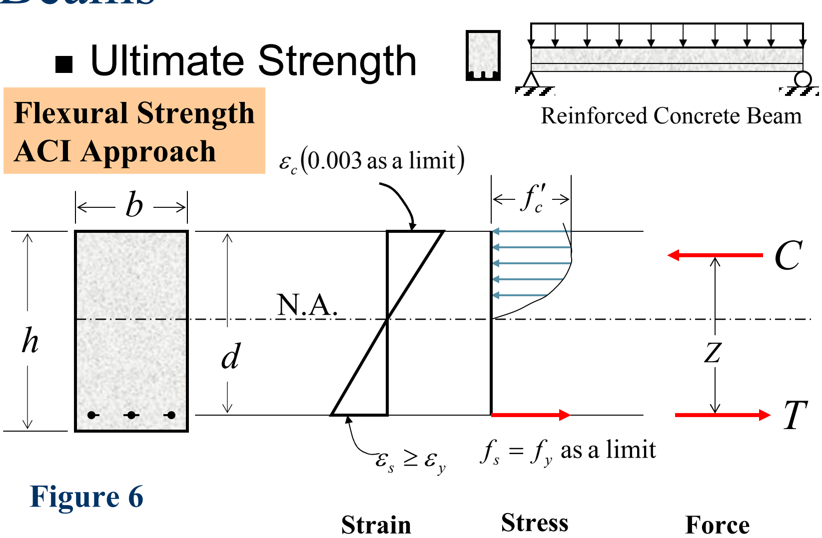


Figure 6

Flexural Strength of Rectangular Beams

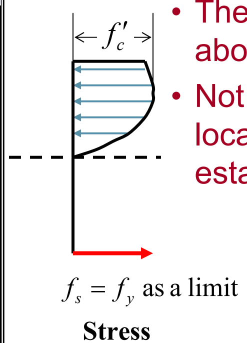
- Possible Values for Concrete Strains due to Loading (Modes of Failure)
  1. Concrete compressive strain is less than 0.003 in./in. when the maximum tensile steel unit equal its yield stress f_y as a limit.
  2. Maximum compressive concrete strain equals 0.003 in./in. and the tensile steel unit stress is less than its yield stress f_y.

Flexural Strength of Rectangular Beams

- Nominal Moment Strength
  - The forces *C* and *T*, and the distance *Z* separated them constitute an internal resisting couple whose maximum value is termed ***nominal moment*** *strength* of the bending member.
  - As a limit, this nominal strength must be capable of resisting the actual design bending moment induced by the applied loads.

Flexural Strength of Rectangular Beams

- Nominal Moment Strength (cont'd)
  - The determination of the moment strength is complex because of
    - The shape of the compressive stress diagram above the neutral axis
    - Not only is *C* difficult to evaluate but also its location relative to the tensile steel is difficult to establish



Flexural Strength of Rectangular Beams

- How to Determine the Moment Strength of Reinforced Concrete Beam?
  - To determine the moment capacity, it is necessary only to know
    1. **The total resultant compressive force *C* in the concrete, and**
    2. **Its location from the outer compressive fiber, from which the distance *Z* may be established.**

Flexural Strength of Rectangular Beams

- How to Determine the Moment Strength of Reinforced Concrete Beam? (cont'd)
  - These two values may easily be established by replacing the unknown complex compressive stress distribution by a fictitious (equivalent) one of simple geometrical shape (e.g., rectangle).
  - Provided that the fictitious distribution results in the same total *C* applied at the same location as in the actual distribution when it is at the point of failure.

The Equivalent Rectangular Block

- Any complicated function can be replaced with an equivalent or fictitious one to make the calculations simple and will give the same results.
- For purposes of simplification and practical application, a fictitious but equivalent rectangular concrete stress distribution was proposed.

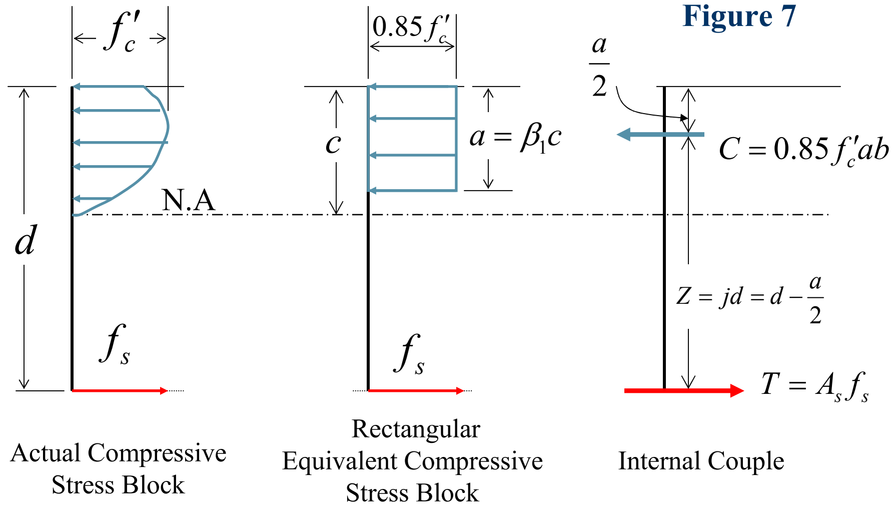
The Equivalent Rectangular Block

- This rectangular stress distribution was proposed by Whiney (1942) and subsequently adopted by the ACI Code
- The ACI code also stipulates that other compressive stress distribution shapes may be used provided that they are in agreement with test results.
- Because of its simplicity, however, the rectangular shape has become the more widely stress distribution (Fig. 7).

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution

Figure 7



The Equivalent Rectangular Block

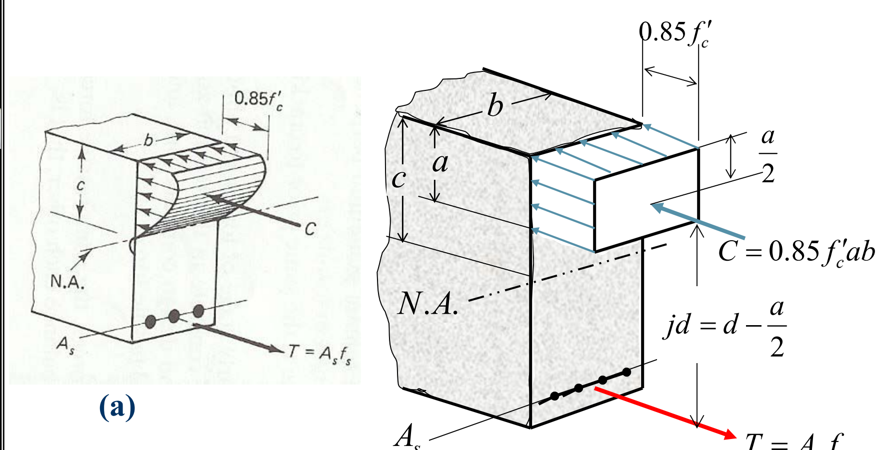
- Whitney's Rectangular Stress Distribution
  - According to Fig. 7, the average stress distribution is taken as

$$\text{Average Stress} = 0.85 f_c' \qquad (2)$$

  - It is assumed to act over the upper area on the beam cross section defined by the width b and a depth a as shown in Fig. 8.

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution



(a)

Figure 8

(b)

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution
  - The magnitude of *a* may determined by

$$a = \beta_1 c \qquad (3)$$

Where

c = distance from the outer fiber to the neutral axis

β_1 = a factor dependent on concrete strength, and is given by

$$\beta_1 = \begin{cases} 0.85 & \text{for } f'_c \leq 4{,}000 \text{ psi} \\ 1.05 - 5 \times 10^{-5} f'_c & \text{for } 4{,}000 \text{ psi} < f'_c \leq 8{,}000 \text{ psi} \\ 0.65 & \text{for } f'_c > 8{,}000 \text{ psi} \end{cases} \qquad (4)$$

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution
  - Using all preceding assumptions, the stress distribution diagram shown in Fig. 8a can be redrawn in Fig. 8b.
  - Therefore, the compressive force *C* can be written as

$$0.85 f'_c ba$$

  - That is, the volume of the compressive block at or near the ultimate when the tension steel has yielded $\varepsilon_s > \varepsilon_y$.

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution
  - The tensile force *T* can be written as $A_s f_y$. Thus equilibrium suggests *C* = *T*, or

$$0.85 f'_c ba = A_s f_y \qquad (5)$$

  - or

$$a = \frac{A_s f_y}{0.85 f'_c b} \qquad (6)$$

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution
  - The moment of resistance of the section, that is, the nominal strength M_n can be expressed as

$$M_n = \left(A_s f_y\right) jd \quad \text{or} \quad M_n = \left(0.85 f'_c ba\right) jd \qquad (7)$$

  - Using Whitney's rectangular block, the lever arm is

$$jd = d - \frac{a}{2} \qquad (8)$$

  - Hence, the nominal resisting moment becomes

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) \quad \text{or} \quad M_n = 0.85 f'_c ba \left(d - \frac{a}{2}\right) \qquad (9)$$

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution
  – If the reinforcement ratio $\rho = A_s/bd$, Eq. 6 can be rewritten as

$$a = \frac{\rho d f_y}{0.85 f'_c} \qquad (10)$$

  – If $r = b/d$, Eq. 9 becomes

$$M_n = \rho r d^2 f_y \left(d - \frac{\rho d f_y}{1.7 f'_c} \right) \qquad (11)$$

The Equivalent Rectangular Block

- Whitney's Rectangular Stress Distribution
  – or

$$M_n = \left[\omega r f'_c (1 - 0.59\omega) d^3 \right] \qquad (12)$$

  – where $\omega = \rho f_y / f_c$`. Eq. 12 sometimes expressed as

$$M_n = Rbd^2 \qquad (13)$$

  – where

$$R = \omega f'_c (1 - 0.59\omega) \qquad (14)$$

Balanced, Overreinforced, and Underreinforced Beams

- Strain Distribution

Figure 9

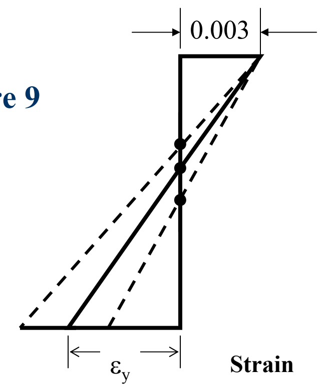
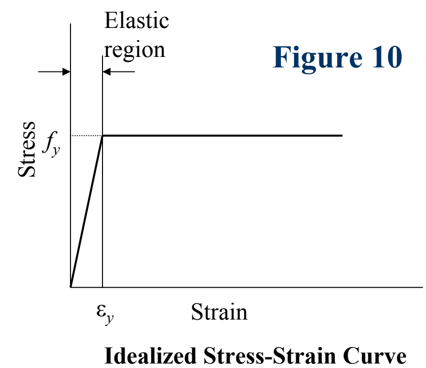


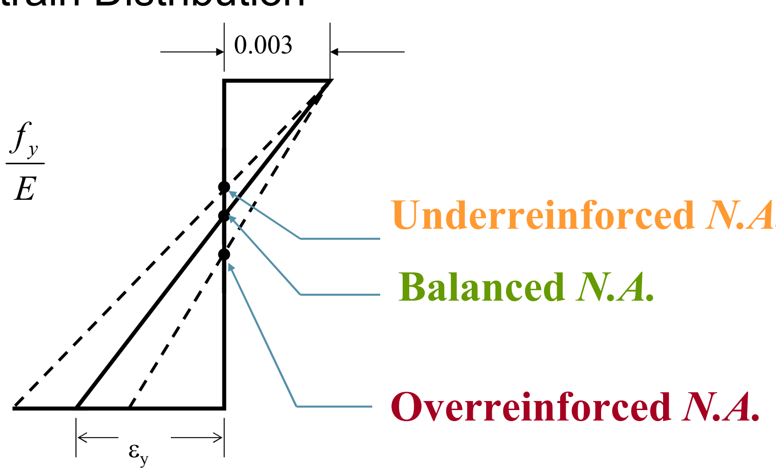
Figure 10



Balanced, Overreinforced, and Underreinforced Beams

- Strain Distribution

$$\varepsilon_y = \frac{f_y}{E}$$



Balanced, Overreinforced, and Underreinforced Beams

- Balanced Condition:

  $\varepsilon_s = \varepsilon_y$ and $\varepsilon_c = 0.003$

- Overreinforced Beam

  $\varepsilon_s < \varepsilon_y$, and $\varepsilon_c = 0.003$. The beam will have more steel than required to create the balanced condition. This is not preferable since will cause the concrete to crush suddenly before that steel reaches its yield point.

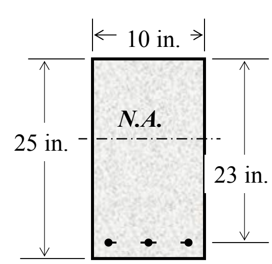
- Underreinforced Beam

  $\varepsilon_s > \varepsilon_y$, and $\varepsilon_c = 0.003$. The beam will have less steel than required to create the balanced condition. This is preferable and is ensured by the ACI Specifications.

Balanced, Overreinforced, and Underreinforced Beams

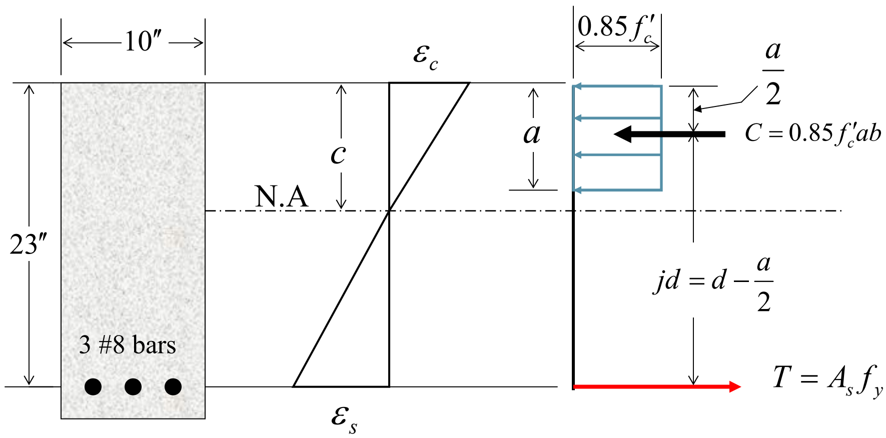
- Example 3

  Determine the nominal moment M_n for a beam of cross section shown, where f'_c = 4,000 psi. Assume A615 grade 60 steel that has a yield strength of 60 ksi and a modulus of elasticity = 29 × 10^6 psi. Is the beam under-reinforced, over-reinforced, or balanced?



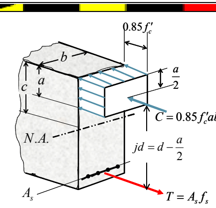
Balanced, Overreinforced, and Underreinforced Beams

- Example 3 (cont'd)



Balanced, Overreinforced, and Underreinforced Beams

- Example 3 (cont'd)



Area for No. 8 bar $= 0.79 \text{ in}^2$ (see Table 1)

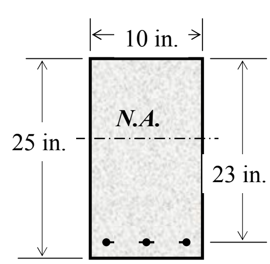
Therefore, $A_s = 3(0.79) = 2.37 \text{ in}^2$ (Also see Table 4.2(a) Text)

Assume that f_y for steel exists subject to later check.

$$C = T$$

$$0.85 f'_c ab = A_s f_y$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{2.37(60)}{0.85(4)(10)} = 4.18 \text{ in.}$$



Balanced, Overreinforced, and Underreinforced Beams

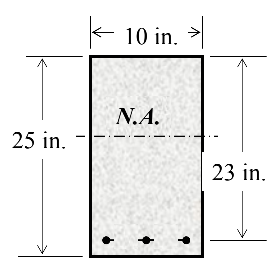
Table 1. ASTM Standard - English Reinforcing Bars

Bar Designation	Diameter in	Area in^2	Weight lb/ft
#3 [#10]	0.375	0.11	0.376
#4 [#13]	0.500	0.20	0.668
#5 [#16]	0.625	0.31	1.043
#6 [#19]	0.750	0.44	1.502
#7 [#22]	0.875	0.60	2.044
#8 [#25]	1.000	0.79	2.670
#9 [#29]	1.128	1.00	3.400
#10 [#32]	1.270	1.27	4.303
#11 [#36]	1.410	1.56	5.313
#14 [#43]	1.693	2.25	7.650
#18 [#57]	2.257	4.00	13.60

Note: Metric designations are in brackets

Balanced, Overreinforced, and Underreinforced Beams

- Example 3 (cont'd)
  - Calculation of M_n



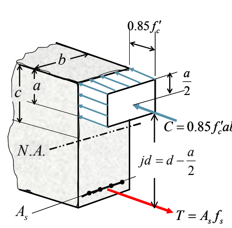
$$M_n = C\left(d - \frac{a}{2}\right) = T\left(d - \frac{a}{2}\right)$$

$$M_n = 0.85 f_c' ab\left(d - \frac{a}{2}\right) = A_s f_y\left(d - \frac{a}{2}\right)$$

Based on steel:

$$M_n = 2.37(60)\left(23 - \frac{4.18}{2}\right) = 2{,}973.4 \text{ in.-kips}$$

$$= \frac{2{,}973.4}{12} = \boxed{247.8 \text{ ft-kips}}$$



Balanced, Overreinforced, and Underreinforced Beams

- Example 3 (cont'd)
  - Check if the steel reaches its yield point before the concrete reaches its ultimate strain of 0.003:
    - Referring to the next figure (Fig. 11), the neutral axis can be located as follows:

Using Eqs. 3 and 4:

$$\beta_1 = 0.85$$

$$a = \beta_1 c$$

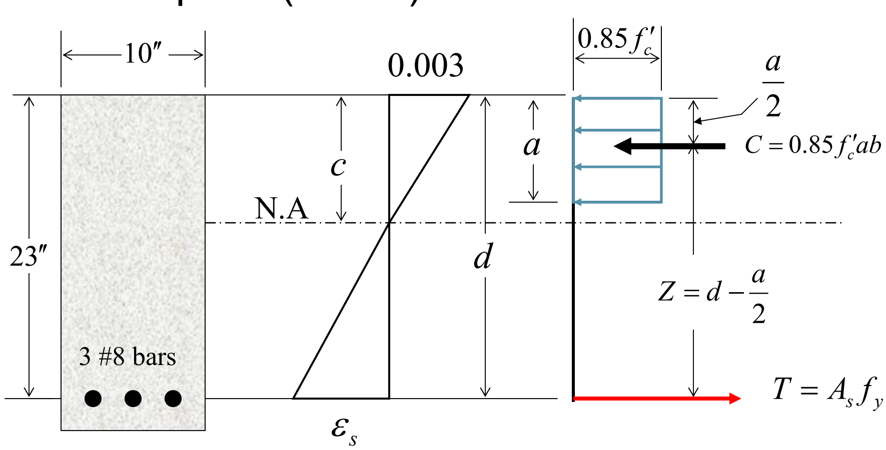
Therefore,

$$c = \frac{a}{\beta_1} = \frac{4.18}{0.85} = 4.92 \text{ in.}$$

Balanced, Overreinforced, and Underreinforced Beams

- Example 3 (cont'd)

Figure 11



Balanced, Overreinforced, and Underreinforced Beams

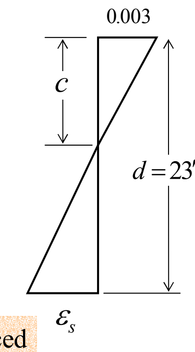
- Example 3 (cont'd)
  - By similar triangles in the strain diagram, the strain in steel when the concrete strain is 0.003 can be found as follows:

$$\frac{0.003}{c} = \frac{\varepsilon_s}{d-c}$$

$$\varepsilon_s = 0.003\frac{d-c}{c} = 0.003\frac{23-4.92}{4.92} = 0.011 \text{ in./in.}$$

The strain at which the steel yields is

$$\varepsilon_y = \frac{f_y}{E_s} = \frac{60{,}000}{29\times10^6} = 0.00207 \text{ in./in.}$$



Since $\varepsilon_s (= 0.011) > \varepsilon_y (= 0.00207)$, the beam is under-reinforced