Flexural Strength of Rectangular Beams

- Ultimate Moment (Strength)
  - The ultimate moment for a reinforced concrete beam can be defined as the moment that exists just prior to the failure of the beam.
  - In order to evaluate this moment, we have to examine the strains, stresses, and forces that exist in the beam.
  - The beam of Fig. 1 has a width of $b$, an effective depth $d$, and is reinforced with a steel area $A_s$. 
Flexural Strength of Rectangular Beams

- **Ultimate Strength**

*Flexural Strength ACI Approach*  
\[ \varepsilon_c (0.003 \text{ as a limit}) \]

**Figure 1**

Strain Stress Force

- **Possible Values for Concrete Strains due to Loading (Modes of Failure)**
  1. Concrete compressive strain is less than 0.003 in./in. when the maximum tensile steel unit equal its yield stress \( f_y \) as a limit.
  2. Maximum compressive concrete strain equals 0.003 in./in. and the tensile steel unit stress is less than its yield stress \( f_y \).
Flexural Strength of Rectangular Beams

**Notes on Concrete Compressive Stresses**
- The ultimate compressive stress for concrete does not occur at the outer fiber.
- The shape of the curve is not the same for different-strength concretes.
- The shape of the curve will also depend on the size and dimensions of the beam.
- The ultimate compressive stress of concrete develops at some intermediate level near, but not at, the extreme outer fiber.

**Nominal Moment Strength**
- The forces $N_C$ and $N_T$, and the distance $Z$ separated them constitute an internal resisting couple whose maximum value is termed *nominal moment strength* of the bending member.
- As a limit, this nominal strength must be capable of resisting the actual design bending moment induced by the applied loads.
Flexural Strength of Rectangular Beams

Nominal Moment Strength (cont’d)

- The determination of the moment strength is complex because of
  - The shape of the compressive stress diagram above the neutral axis
  - Not only is $N_C$ difficult to evaluate but also its location relative to the tensile steel is difficult to establish

\[ f_s = f_y \] as a limit

Stress

How to Determine the Moment Strength of Reinforced Concrete Beam?

- To determine the moment capacity, it is necessary only to know
  1. The total resultant compressive force $N_C$ in the concrete, and
  2. Its location from the outer compressive fiber, from which the distance $Z$ may be established.
Flexural Strength of Rectangular Beams

- How to Determine the Moment Strength of Reinforced Concrete Beam? (cont’d)
  - These two values may easily be established by replacing the unknown complex compressive stress distribution by a fictitious (equivalent) one of simple geometrical shape (e.g., rectangle).
  - Provided that the fictitious distribution results in the same total \( N_C \) applied at the same location as in the actual distribution when it is at the point of failure.

**Mathematical Motivation**

- Consider the function
  \[
  f(x) = y = 2\sqrt{x}
  \]  
  (1)
- Plot of this function is shown in Fig. 2 for \( x \) ranges from 0 to 4, and \( y \) from 0 to 4.
- The area under the curve will be determined analytically.
- Note that in real situation this area will be the equivalent, for example, to compressive force \( N_C \) for concrete per unit length.
Flexural Strength of Rectangular Beams

Mathematical Motivation (cont’d)

Area under the Curve

\[ A = \frac{4}{2}\int y \, dx = \frac{4}{2}\left(2\sqrt{x}\right) \, dx = 2\frac{4}{2}\int x^\frac{1}{2} \, dx = 10.7 \text{ in}^2 \]

\[ x = \frac{\int x \, dA}{\int dA} = 10.7 \]

\[ \int x \, dA = \frac{4}{4}\int_0^4 x(y \, dx) = 25.6 \]

\[ \frac{25.6}{10.7} = 2.4 \text{ in.} \]
Flexural Strength of Rectangular Beams

- Mathematical Motivation (cont’d)
  - Objective
    - Our objective is to find a fictitious or equivalent curve results in the same total area $A$ applied at the same location as the actual curve.
    - Find $x'$ and $y'$

Actual Curve

Equivalent Simple Curve

Calculations of $x'$ and $y'$

$$x' = 4 - \bar{x} = 4 - 2.4 = 1.6 \text{ in.}$$

Area = $2x'y'$

$$y' = \frac{\text{Area}}{2x'} = \frac{10.7}{2(1.6)} = 3.34 \text{ in.}$$
Flexural Strength of Rectangular Beams

Mathematical Motivation (cont'd)

- If we are dealing with a concrete compressive stress distribution and we let \( x' = a / 2 \), then

\[
y' = 0.84 f'_c
\]

and

\[
a = 2x' = \beta_c = 2(1.6) = 3.2 \text{ in.}
\]

Then,

\[
\beta_c = \frac{3.2}{4} = 0.80
\]

Equivalent Stress Distribution

As we saw in our previous mathematical example, any complicated function can be replaced with an equivalent or fictitious one to make the calculations simple and will give the same results.

For purposes of simplification and practical application, a fictitious but equivalent rectangular concrete stress distribution was proposed.
This rectangular stress distribution was proposed by Whiney (1942) and subsequently adopted by the ACI Code.

The ACI code also stipulates that other compressive stress distribution shapes may be used provided that they are in agreement with test results.

Because of its simplicity, however, the rectangular shape has become the more widely used stress distribution (Fig. 2).

**Whitney’s Rectangular Stress Distribution**

\[ N_C = 0.85 f'_{c',ab} \]

\[ Z = d - \frac{a}{2} \]

\[ N_C = A_s f_y \]
Whitney’s Rectangular Stress Distribution

According to Fig. 2, the average stress distribution is taken as

Average Stress = \( 0.85 f'_c \)

It is assumed to act over the upper area on the beam cross section defined by the width \( b \) and a depth \( a \) as shown in Fig. 3.
Whitney’s Rectangular Stress Distribution
– The magnitude of \( a \) may determined by

\[
a = \beta_1 c
\]

(2)

Where

- \( C \) = distance from the outer fiber to the neutral axis
- \( \beta_1 \) = a factor dependent on concrete strength, and is given by

\[
\beta_1 = \begin{cases} 
0.85 & \text{for } f'_c \leq 4,000 \text{ psi} \\
1.05 - 5 \times 10^{-3} f'_c & \text{for } 4,000 \text{ psi} < f'_c \leq 8,000 \text{ psi} \\
0.65 & \text{for } f'_c > 8,000 \text{ psi}
\end{cases}
\]

(3)

Example 1

Determine the nominal moment \( M_n \) for a beam of cross section shown, where \( f'_c = 4,000 \) psi.
Assume A615 grade 60 steel that has a yield strength of 60 ksi and a modulus of elasticity = \( 29 \times 10^6 \) psi.
Example 1 (cont’d)

Area for No. 8 bar = 0.79 in² (see Table 1)

Therefore, \( A_s = 3(0.79) = 2.37 \text{ in}^2 \) \( \text{(Also see Table A-2 Text)} \)

Assume that \( f_y \) for steel exists subject later check.

\[
N_c = N_s \\
0.85 f'_c ab = A_s f_y \\
a = \frac{A_s f_y}{0.85 f'_c b} = \frac{2.37(60)}{0.85(4)(10)} = 4.18 \text{ in.}
\]
Table 1. ASTM Standard - English Reinforcing Bars

<table>
<thead>
<tr>
<th>Bar Designation</th>
<th>Diameter in</th>
<th>Area in²</th>
<th>Weight lb/ft</th>
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<td>0.668</td>
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<td>#6 [#19]</td>
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Note: Metric designations are in brackets

Example 1 (cont'd)
Calculation of $M_n$

$$M_n = N_C \left(d - \frac{a}{2}\right) = N_T \left(d - \frac{a}{2}\right)$$

$$M_n = 0.85 f'_{ab} \left(d - \frac{a}{2}\right) = A_s f_y \left(d - \frac{a}{2}\right)$$

Based on steel:

$$M_n = 2.37(60\left(23 - \frac{4.18}{2}\right)) = 2,973.4 \text{ in.-kips}$$

$$= \frac{2,973.4}{12} = 247.8 \text{ ft-kips}$$
Example 1 (cont’d)

Check if the steel reaches its yield point before the concrete reaches its ultimate strain of 0.003:

- Referring to the next figure (Fig. 4), the neutral axis can be located as follows:

Using Eqs. 2 and 3:

$$\beta_i = 0.85$$

$$a = \beta_i c$$

Therefore,

$$c = \frac{a}{\beta_i} = \frac{4.18}{0.85} = 4.92\text{ in.}$$
**Example 1 (cont’d)**

By similar triangles in the strain diagram, the strain in steel when the concrete strain is 0.003 can be found as follows:

\[ \frac{0.003}{c} = \frac{\varepsilon_s}{d - c} \]

\[ \varepsilon_s = 0.003 \frac{d - c}{c} = 0.003 \frac{23 - 4.92}{4.92} = 0.011 \text{ in./in.} \]

The strain at which the steel yields is

\[ \varepsilon_y = \frac{f_y}{E_s} = \frac{60,000}{29 \times 10^6} = 0.00207 \text{ in./in.} \]

Since \( \varepsilon_s (= 0.011) > \varepsilon_y (= 0.00207) \) **OK**
Balanced, Overreinforced, and Underreinforced Beams

- **Balanced Condition:**
  \[ \varepsilon_s = \varepsilon_y \quad \text{and} \quad \varepsilon_c = 0.003 \]

- **Overreinforced Beam**
  \[ \varepsilon_s < \varepsilon_y, \quad \text{and} \quad \varepsilon_c = 0.003 \]
  The beam will have more steel than required to create the balanced condition. This is not preferable since will cause the concrete to crush suddenly before that steel reaches its yield point.

- **Underreinforced Beam**
  \[ \varepsilon_s > \varepsilon_y, \quad \text{and} \quad \varepsilon_c = 0.003 \]
  The beam will have less steel than required to create the balanced condition. This is preferable and is ensured by the ACI Specifications.
Reinforcement Ratio Limitations and Guidelines

- Although failure due yielding of the steel is gradual with adequate warning of collapse, failure due to crushing of the concrete is sudden and without warning.
- The first type (Underreinforced beam) is preferred and ensured by the specifications of the ACI.
- The ACI code stipulates that
  \[ A_s \leq 0.75 A_{sb} \]  

Steel Ratio

- The steel ratio (sometimes called reinforcement ratio) is given by
  \[ \rho = \frac{A_s}{bd} \]  

ACI stipulates that
  \[ \rho_{\text{max}} = 0.75 \rho_b \]  
  or  \[ A_{s_{\text{max}}} = 0.75 A_{sb} \]
Reinforcement Ratio Limitations and Guidelines

**Example 2**

Determine the amount of steel required to create a balanced condition for the beam shown, where $f_y = 4,000$ psi. Assume A615 grade 60 steel that has a yield strength of 60 ksi and a modulus of elasticity $E_y = 29 \times 10^6$ psi. Also check the code requirement for ductile-type beam.

Area for No. 8 bar = 0.79 in$^2$ (see Table 1)

Therefore, $A_s = 3(0.79) = 2.37$ in$^2$

The strain at which the steel yields is

$$
\varepsilon_y = \frac{f_y}{E_y} = \frac{60,000}{29 \times 10^6} = 0.00207 \text{ in./in.}
$$

In reference to the strain diagram of Fig. 7, and from similar triangles,

$$
\frac{c_b}{0.003} = \frac{d - c_b}{0.00207}
$$
Reinforcement Ratio Limitations and Guidelines

Table 1. ASTM Standard - English Reinforcing Bars

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Note: Metric designations are in brackets

Example 2 (cont’d)

Figure 7

\[ N_A = A_y f_y \]
Reinforcement Ratio Limitations and Guidelines

■ Example 2 (cont’d)

\[ \frac{c_b}{0.003} = \frac{23 - c_b}{0.00207} \]

From which, \( c_b = 13.6 \) in.

Using Eqs. 2 and 3:

\[ \beta_i = 0.85 \text{ because } f_i' = 4,000 \text{ psi} \]

\[ a = \beta_i c = 0.85(13.6) = 11.6 \text{ in.} \]

Therefore,

\[ A_{sb} = \frac{N_{Ch}}{f_y} = \frac{394.4}{60} = 6.57 \text{ in}^2 \]

Hence, required steel for balanced condition \( = 6.57 \text{ in}^2 \)

From Eq. 6,

\[ A_{max} = 0.75A_{sb} = 0.74(6.57) = 4.93 \text{ in}^2 > A_s = 2.37 \text{ in}^2 \text{ OK} \]
Reinforcement Ratio Limitations and Guidelines

Steel Ratio Formula for Balanced Beam

Instead of using laborious techniques for determining the balanced steel of beam, the following formula can be used to determine the steel ratio $\rho_b$ at the balance condition:

$$\rho_b = \frac{0.85 f'_c \beta_1}{f_y} \left( \frac{87,000}{f_y + 87,000} \right)$$  

(7)

where

- $f'_c$ = compressive strength of concrete (psi)
- $f_y$ = yield strength of steel (psi)
- $\beta_1$ = factor that depends on $f'_c$ as given by Eq. 3

Lower Limit for Steel Reinforcement

- The ACI Code establishes a lower limit on the amount of tension reinforcement. The code states that where tensile reinforcement is required, the steel area $A_s$ shall not be less than that given by

$$A_{s,\text{min}} = \frac{3f'_c}{f_y} b_w d \geq \frac{200}{f_y} b_w d$$  

(8)

Note that for rectangular beam $b_w = b$
## Reinforcement Ratio Limitations and Guidelines

(Table A-5 Text)

### Table 1. Design Constants

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<thead>
<tr>
<th>$f_y$ (ksi)</th>
<th>$\frac{\sqrt{f_y}}{f_y} \leq 200 \frac{f_y}{f_y'}$</th>
<th>$\rho_{\text{min}} = 0.75 \rho_b$</th>
<th>Recommended Design Values</th>
<th>$\rho_b$</th>
<th>$\bar{e}$ (ksi)</th>
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