CHAPTER 8e. DIFFERENTIAL EQUATIONS

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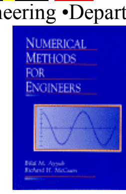

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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First-order Ordinary Differential Equations

- Galerkin Method

  Procedure for Deriving Galerkin Approximating Polynomial

  1. For a given ordinary differential equation, assume that the solution is a differentiable function such as given by the following polynomial:

$$\hat{y} = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \qquad (29)$$

First-order Ordinary Differential Equations

- Galerkin Method
  2. Use the boundary condition of the ordinary differential equation to evaluate one of the coefficients of the assumed function or to provide a condition toward the solution of the coefficients.
  3. Provide an expression for the error e as

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx} \qquad (30)$$

First-order Ordinary Differential Equations

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- Galerkin Method
  4. Compute the weight w_i for each coefficient, where

$$w_i = \frac{d\hat{y}}{db_i} \qquad (31)$$

  5. The normal equations are computed from

$$\int_x w_i e \, dx = 0 \qquad \text{for } i = 1, 2, \cdots, n \qquad (32)$$

First-order Ordinary Differential Equations

- Example 15 – Galerkin Method

  Using the Galerkin method with a linear model (i.e. $\hat{y} = b_o + b_1 x$) and for $0 \le x \ge 1$ to find an approximating function for the following differential:

$$\frac{dy}{dx} = xy \quad \text{such that } y = 1 \text{ at } x = 0$$

  Using this function, estimate y at $x = 0.6$ and compare with the true value.

First-order Ordinary Differential Equations

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- Example 15 (cont'd) – Galerkin Method

$$\hat{y} = b_0 + b_1 x$$

$$\hat{y}(0) = 1 = b_0 + b_1(0) \Rightarrow b_0 = 1$$

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx}$$

$$\hat{y} = 1 + b_1 x \qquad \text{and} \qquad \frac{d\hat{y}}{dx} = b_1$$

First-order Ordinary Differential Equations

- Example 15 (cont'd) – Galerkin Method

$$\frac{dy}{dx} = xy \qquad \text{(given by the DE)}$$

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx} = b_1 - xy = b_1 - x(1 - b_1 x)$$

$$w_1 = \frac{d\hat{y}}{db_1} = x$$

First-order Ordinary Differential Equations

- Example 15 (cont'd) – Galerkin Method

$$\int_0^1 w_1 e\, dx = \int_0^1 x\left[b_1 - x(1 - b_1 x)\right] dx = 0$$

$$\int_0^1 \left[x b_1 - x^2 + b_1 x^3\right] dx = 0$$

$$\left.\frac{x^2 b_1}{2} - \frac{x^3}{3} + \frac{x^4 b_1}{4}\right|_0^1 = \frac{b_1}{2} - \frac{1}{3} + \frac{b_1}{4} = 0$$

$$\frac{3b_1}{4} = \frac{1}{3} \Rightarrow b_1 = \frac{4}{9}$$

First-order Ordinary Differential Equations

- Example 15 (cont'd) – Galerkin Method

Therefore the approximating function (linear polynomial) is given by

$$\hat{y} = b_0 + b_1 x = 1 + \frac{4}{9} x$$

and $\hat{y}(0.6) = 1 + \frac{4}{9}(0.6) = 1.2667$

The true value is given by

$$y(0.6) = e^{x^2/2} = e^{(0.6)^2/2} = 1.1972$$

$$\text{error (\%)} = \left| \frac{1.1972 - 1.2667}{1.1972} \right| \times 100 = 5.8\%$$

Higher-order Differential Equations

- Many engineering problems require the solution of higher-order differential equations.
- For example a second-order differential equation can be given as

$$\frac{d^2 y}{dx^2} = f\left(x, y, \frac{dy}{dx} \right) \qquad (33)$$

Higher-order Differential Equations

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- The function $f(.)$ does not need to include all the parameters $x, y,$ and dy/dx.
- Examples of higher-order differential equations are

$$\frac{d^2y}{dx^2} = 2x \quad \text{such that } y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0$$

$$\frac{d^3y}{dx^3} - 2x - yx + 4 = 0$$

Higher-order Differential Equations

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- The methods previously introduced can be used to solve higher-order differential equations after transforming them into systems of first-order differential equations.
- The procedure for transforming the DE's for the following example is as follows:

Higher-order Differential Equations

$$\frac{d^2 y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

becomes

$$\frac{dy_2}{dx} = f\left(x, y, y_2\right)$$

$$\frac{dy_1}{dx} = y_2 \qquad \text{where } y_1 = y_2$$

$$y_2 = \frac{dy_1}{dx} = \frac{dy}{dx}$$

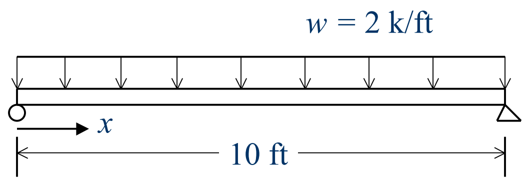
Higher-order Differential Equations

- Higher-order Differential Equations

$$\begin{aligned}
\frac{dy_1}{dx} &= f_1\left(x, y_1, y_2, \cdots, y_n\right) \\
\frac{dy_2}{dx} &= f_1\left(x, y_1, y_2, \cdots, y_n\right) \\
&\vdots \\
\frac{dy_n}{dx} &= f_n\left(x, y_1, y_2, \cdots, y_n\right)
\end{aligned} \qquad (34)$$

Higher-order Differential Equations

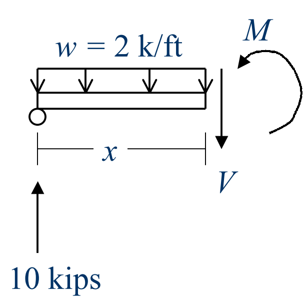
- Example – Simply Supported Beam



$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

Higher-order Differential Equations

- Example (cont'd) – Simply Supported Beam



$$\sum M = -M + 10x - 2x\left(\frac{x}{2}\right) = 0$$

$$-M + 10x - x^2 = 0$$

$$M = 10x - x^2$$

Higher-order Differential Equations

- Example – Simply Supported Beam

Let

$$\theta = \frac{dy}{dx} = \text{rotation}$$

$$\frac{d^2 y}{dx^2} = \frac{d\theta}{dx} = \frac{M(x)}{EI}$$

Hence

$$\frac{d\theta}{dx} = \frac{10x - x^2}{EI}$$

$$\frac{dy}{dx} = \theta$$

Higher-order Differential Equations

- Example (cont'd) – Simply Supported Beam

Assume that EI = 3600 kip/ft^2 and at $x = 0$, $y = 0$ and θ = - 0.02314, and h = 0.1.

Using basic Euler's method for example, the following equations result:

$$\theta_{i+1} = \theta_i + f_\theta(x_i, y_i, \theta_i)h$$

$$y_{i+1} = y_i + f_y(x_i, y_i, \theta_i)h$$

Higher-order Differential Equations

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- Example (cont'd) – Simply Supported Beam

First Iteration (i = 0)

$$x_0 = 0, \theta_0 = -0.02314, \text{ and } y = 0$$

$$f_\theta(x_i, \theta_i, y_i) = f_\theta(x_0, \theta_{0i}, y_0) = \left.\frac{d\theta}{dx}\right|_{\substack{x=x_0\\ \theta=\theta_0\\ y=y_0}} = \frac{10x - x^2}{EI}$$

$$f_\theta(0, -0.0231481, 0) = \frac{10(0) - (0)^2}{3600} = 0$$

$$\theta_1 = \theta_0 + f_\theta(0, -0.0231481, 0)h$$

$$\theta_1 = -0.0231481 + 0 = -0.0231481$$

Higher-order Differential Equations

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- Example (cont'd) – Simply Supported Beam

$$f_y(x_i, \theta_i, y_i) = f_y(x_0, \theta_0, y_0)h = \frac{dy}{dx} = \theta_0$$

$$f_y(0, -0.0231481, 0) = \theta_0 = -0.0231481$$

$$y_1 = y_0 + f_y(x_0, \theta_0, y_0)h$$

$$= y_0 + f_y(0, -0.0231481, 0)$$

$$y_1 = 0 - 0.0231481(0.1) = -0.00231481$$

Higher-order Differential Equations

- Example (cont'd) – Simply Supported Beam

  Second Iteration (i = 1)

$$x_1 = 1.1, \theta_1 = -0.02314, \text{ and } y = -0.00231481$$

$$f_\theta(x_i, \theta_i, y_i) = f_\theta(x_1, \theta_{1i}, y_1) = \left.\frac{d\theta}{dx}\right|_{\substack{x=x_1\\ \theta=\theta_1\\ y=y_1}} = \frac{10x - x^2}{EI}$$

$$f_\theta(0.1, -0.0231481, -0.00231481) = \frac{10(0.1) - (0.1)^2}{3600} = 0.00025$$

$$\theta_2 = \theta_1 + f_\theta(0.1, -0.0231481, -0.00231481)h$$

$$\theta_2 = -0.0231481 + 0.000275(0.1) = -0.0231206$$

Higher-order Differential Equations

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- Example (cont'd) – Simply Supported Beam

$$f_y(x_i, \theta_i, y_i) = f_y(x_1, \theta_1, y_1)h = \frac{dy}{dx} = \theta_1$$

$$f_y(0.1, -0.0231206, 0) = \theta_1 = -0.0231206$$

$$y_2 = y_1 + f_y(x_1, \theta_1, y_1)h$$

$$= y_1 + f_y(0, -0.0231206, 0)$$

$$y_2 = -0.00231481 - 0.0231206(0.1) = -0.0046269$$

Higher-order Differential Equations

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- Example (cont'd) – Simply Supported Beam

The exact solution is given by

$$\theta = \left(5x^3 - \frac{x^3}{3}\right)\frac{1}{3600} - 0.0231481$$

$$y = \left(\frac{5x^3}{3} - \frac{x^4}{12}\right)\frac{1}{3600} - 0.0231481x$$

x (ft)	$\frac{d\theta}{dx}$	$\theta = \frac{dy}{dx}$	y (ft)	Exact θ	Exact y (ft)
0	0	-0.0231481	0	-0.0231481	0
0.1	0.0002750	-0.0231481	-0.0023148	-0.0231343	-0.0023143
0.2	0.0005444	-0.0231206	-0.0046296	-0.0230933	-0.0046260
0.3	0.0008083	-0.0230662	-0.0069417	-0.0230256	-0.0069321
0.4	0.0010667	-0.0229853	-0.0092483	-0.0229318	-0.0092302
0.5	0.0013194	-0.0228787	-0.0115468	-0.0228125	-0.0115176
0.6	0.0015667	-0.0227467	-0.0138347	-0.0226681	-0.0137919
0.7	0.0018083	-0.0225900	-0.0161094	-0.0224993	-0.0160504
0.8	0.0020444	-0.0224092	-0.0183684	-0.0223066	-0.0182909
0.9	0.0022750	-0.0222048	-0.0206093	-0.0220906	-0.0205110
1	0.0025000	-0.0219773	-0.0228298	-0.0218518	-0.0227083
2	0.0044444	-0.0185564	-0.0434304	-0.0183333	-0.0429629
3	0.0058333	-0.0134412	-0.0598018	-0.0131481	-0.0588193
4	0.0066667	-0.0071870	-0.0704996	-0.0068518	-0.0688887
5	0.0069444	-0.0003495	-0.0746349	0.0000000	-0.0723377
6	0.0066667	0.0065158	-0.0718744	0.0068519	-0.0688886
7	0.0058333	0.0128533	-0.0624403	0.0131482	-0.0588191
8	0.0044444	0.0181075	-0.0471104	0.0183334	-0.0429626
9	0.0025000	0.0217227	-0.0272179	0.0218519	-0.0227079
10	0.0000000	0.0231436	-0.0046518	0.0231482	0.0000000

Boundary-Value Problems

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- The previous methods for solving ordinary differential equations can be used to solve higher-order differential equations as discussed earlier.
- These methods require the initial conditions to be at the same x value.

Boundary-Value Problems

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- For example, the initial conditions in Example 15 were given as

$$x_0 = 0$$
$$y_0 = 0$$
$$\theta_0 = -0.0231481$$

- The rotation θ is not usually known.
- However, in Ex. 15 it is known that at $x = 0$, $y = 0$ and at $x = 10$, $y = 0$

Boundary-Value Problems

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- The two conditions have different x values. Therefore, they cannot be used in solving the differential equation based on the previous methods.
- In this case, we are dealing with a *boundary-value problem*.
- Boundary-value problems can be solved numerically using *Shooting* and *Finite-difference* methods.

Boundary-Value Problems

- Shooting Method
  - The shooting method is a trial-and-error method that uses any of the previously introduced methods for solving differential equations.
  - The shooting method is based on converting the boundary-value problem into an equivalent initial-value problem.

Boundary-Value Problems

- Finite-Difference Method

  Recall the finite-difference expressions for approximating the first and second derivatives based on two-step method

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{2h} \qquad (35)$$

$$f''(x_i) \approx \frac{f(x_{i-1}) - 2(x_i) + f(x_{i+1})}{h^2} \qquad (36)$$

Boundary-Value Problems

- Finite-Difference Method
  - In this method, the derivatives in the differential equation are replaced by finite-difference equations (formulas).
  - Then, the resulting DE, which is now in a finite-difference form, is used at some interior points using a selected step size h and the boundary conditions.

Boundary-Value Problems

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- Finite-Difference Method
  - Each use of the finite-difference equation results in a linear equation in terms of the unknown solutions at the selected interior points.
  - In this case, we get a system of linear equations that need to be solved simultaneously to obtain the solution at the interior points.

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Boundary-Value Problems

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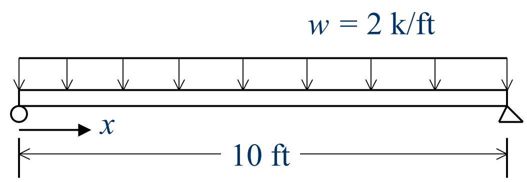
- Example 16 – Simply Supported Beam

  The simply supported beam is subjected to distributed loading w as shown in the figure. Numerically, find the bending moment M at each node along its span.



Boundary-Value Problems

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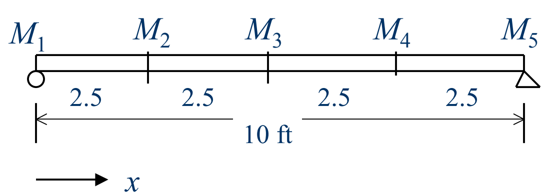
- Example 16 – Simply Supported Beam

  B.C.'s :

$$\text{at } x = 0, M = 0$$
$$\text{and at } x = 10, M = 0$$



Boundary-Value Problems

- Example 16 (cont'd) – Simply Supported Beam

  The governing differential equation in this case is given by

$$\frac{d^2M}{dx^2} = w = \text{distributed load}$$

or

$$\frac{d^2M}{dx^2} = -2$$

Boundary-Value Problems

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- Example 16 (cont'd) – Simply Supported Beam

$$\boxed{f''(x_i) \approx \frac{f(x_{i-1}) - 2(x_i) + f(x_{i+1})}{h^2} \quad \text{(Eq. 36)}}$$

  The original differential equation can be converted to a finite-difference equation (see Eq. 36) as follows:

$$\frac{d^2M}{dx^2} \approx \frac{M_{i-1} - 2M_i + M_{i+1}}{h^2}$$

$$\frac{M_{i-1} - 2M_i + M_{i+1}}{h^2} = -2 \qquad (37)$$

Boundary-Value Problems

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- Example 16 (cont'd) – Simply Supported Beam

  h, is case, is equal to 2.5 ft, therefore Eq. 37 can be simplified as follows.

$$\frac{M_{i-1} - 2M_i + M_{i+1}}{(2.5)^2} = -2$$

or

$$M_{i-1} - 2M_i + M_{i+1} = -2(6.25)$$

$$M_{i-1} - 2M_i + M_{i+1} = -12.5 \qquad (38)$$

Boundary-Value Problems

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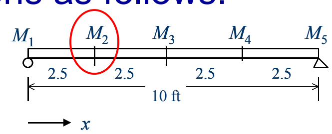
- Example 16 (cont'd) – Simply Supported Beam

  Application of Eq. 38 at the interior nodes 2, 3, and 4, yields a set of three simultaneous equations as follows:

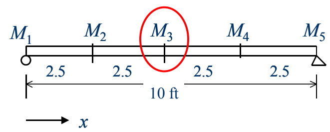
  <u>At Node 2:</u>



$$\overset{0}{\cancel{M_1}} - 2M_2 + M_3 = -12.5$$

Boundary-Value Problems

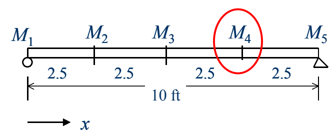
- Example 16 (cont'd) – Simply Supported Beam



At Node 3:

$$M_2 - 2M_3 + M_4 = -12.5$$

At Node 4



$$M_3 - 2M_4 + \cancelto{0}{M_5} = -12.5$$

Boundary-Value Problems

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- Example 16 (cont'd) – Simply Supported Beam

$$\begin{aligned} -2M_2 + M_3 \qquad &= -12.5 \\ M_2 - 2M_3 + M_4 &= -12.5 \\ M_3 - 2M_4 &= -12.5 \end{aligned}$$

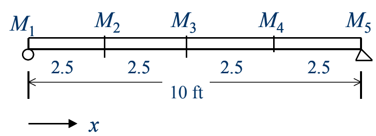
$$\Downarrow$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -12.5 \\ -12.5 \\ -12.5 \end{bmatrix} \qquad (39)$$

Boundary-Value Problems

- Example 16 (cont'd) – Simply Supported Beam

  Equation 39 yields the solution as follows:



$$\begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 18.75 \\ 25.00 \\ 18.75 \end{bmatrix} \text{ft - kips}$$

☺

I wish you good luck in your final ☺ Exams