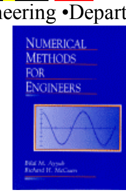
CHAPTER 8c. DIFFERENTIAL EQUATIONS

• A. J. Clark School of Engineering •Department of Civil and Environmental Engineering



by

Dr. Ibrahim A. Assakkaf

Spring 2001

ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

University of Maryland, College Park

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 5 – Euler's Method

  Solve the following differential equation using Euler's method for $1 \leq x \geq 2$ with a step size of *h* = 0.1 :

$$\frac{dy}{dx} = 3x^2 y \qquad \text{such that } y = 1 \text{ at } x = 1$$

$x_0 = 1$

$y_0 = 1$

Here we have

$$y(1) = 1 \qquad \text{or} \qquad x_0 = 1$$
$$y_0 = 1$$

First-order Ordinary Differential Equations

- Example 5 (cont'd) – Euler's Method

  First Iteration ($i = 0$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$x_0 = 1, \quad y_0 = 1, \text{ and } \quad h = 0.1$$
$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{\substack{x_0=1\\y_0=1}} = 3x^2y = 3(1)^2(1) = 3$$
$$\therefore y_1 = 1 + 0.1(3) = 1 + 0.3 = 1.3$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 5 (cont'd) – Euler's Method

  Second Iteration ($i = 1$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$x_1 = 1.1, \quad y_1 = 1.3, \text{ and } \quad h = 0.1$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=1.1\\y_0=1.3}} = 3x^2y = 3(1.1)^2(1.3) = 4.7190$$
$$\therefore y_2 = 1.3 + 0.1(4.7190) = 1.7719$$

First-order Ordinary Differential Equations

■ Example 5 (cont'd) – Euler's Method

Third Iteration ($i = 2$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$x_2 = 1.2, \quad y_2 = 1.7719, \text{ and } \quad h = 0.1$$

$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1 = 1.2 \\ y_0 = 1.7719}} = 3x^2 y = 3(1.2)^2(1.7719) = 7.65461$$

$$\therefore y_3 = 1.7719 + 0.1(7.65461) = 2.53736$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 5 (cont'd) – Euler's Method

- See the spreadsheet output in the next viewgraph for the rest of the iterations.
- Expression for the exact solution can be obtained as follows:

$$\boxed{\begin{array}{l} x_0 = 0 \\ \ln 1 = 0 \end{array}}$$

$$\frac{dy}{dx} = 3x^2 y \Rightarrow \int_1^y \frac{dy}{y} = \int_1^x 3x^2 dx$$

$$\ln y - \ln 1 = \left.\frac{3x^3}{3}\right|_1^x = \left.x^3\right|_1^x \Rightarrow \ln y = x^3 - 1$$

$$y = e^{x^3 - 1}$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 5 (cont'd) – Euler's Method

i	x	x_i	y_i	$f(x_i, y_i)$	y (Euler)	y (True)	% Error
0	1	1	1	3		1	
1	1.1	1.1	1.300000	4.719	1.300000	1.39236	6.63
2	1.2	1.2	1.771900	7.654608	1.771900	2.070935	14.44
3	1.3	1.3	2.537361	12.86442	2.537361	3.310171	23.35
4	1.4	1.4	3.823803	22.48396	3.823803	5.720178	33.15
5	1.5	1.5	6.072199	40.98734	6.072199	10.75101	43.52
6	1.6	1.6	10.1709329	78.11276	10.1709329	22.10934	54.00
7	1.7	1.7	17.9822093	155.9058	17.9822093	50.04887	64.07
8	1.8	1.8	33.5727848	326.3275	33.5727848	125.4616	73.24
9	1.9	1.9	66.2055316	717.0059	66.2055316	350.3736	81.10
10	2	2	137.9061223	1654.873	137.9061223	1096.633	87.42

True Function : $y = e^{x^3 - 1}$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 6 – Euler's Method

Repeat Example 5 for $1 \le x \ge 2$ with a step size of h = 0.05 :

$$\frac{dy}{dx} = 3x^2 y \qquad \text{such that } y = 1 \text{ at } x = 1$$

$x_0 = 1$
$y_0 = 1$

Again, we have

$$y(1) = 1 \qquad \text{or} \qquad \begin{matrix} x_0 = 1 \\ y_0 = 1 \end{matrix}$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 6 (cont'd) – Euler's Method

First Iteration ($i = 0$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$x_0 = 1, \quad y_0 = 1, \text{ and } \quad h = 0.05$$
$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{\substack{x_0=1\\y_0=1}} = 3x^2y = 3(1)^2(1) = 3$$
$$\therefore y_1 = 1 + 0.05(3) = 1 + 0.15 = 1.15$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 6 (cont'd) – Euler's Method

Second Iteration ($i = 1$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$x_1 = 1.05, \quad y_1 = 1.15, \text{ and } \quad h = 0.05$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=1.05\\y_0=1.15}} = 3x^2y = 3(1.05)^2(1.15) = 3.80363$$
$$\therefore y_2 = 1.15 + 0.05(3.80363) = 1.34018$$

First-order Ordinary Differential Equations

- Example 6 (cont'd) – Euler's Method

  Third Iteration ($i = 2$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$x_2 = 1.1, \quad y_2 = 1.34018, \text{ and } \quad h = 0.05$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=1.1\\ y_0=1.34018}} = 3x^2 y = 3(1.1)^2(1.34018) = 4.86485$$
$$\therefore y_3 = 1.34018 + 0.05(4.86485) = 1.58342$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 6 (cont'd) – Euler's Method
  - See the spreadsheet output in the next viewgraph for the rest of the iterations.
  - From Example 5, the expression for the exact solution is

$$y = e^{x^3 - 1}$$

First-order Ordinary Differential Equations

■ Example 6 (cont'd) – Euler's Method

True Function :

$$y = e^{x^3 - 1}$$

$$h = 0.05$$

i	x	x_i	y_i	$f(x_i, y_i)$	y (Euler)	y (True)	% Error
0	1	1	1	3		1	
1	1.05	1.05	1.150000	3.803625	1.150000	1.170727	1.77
2	1.1	1.1	1.340181	4.864858	1.340181	1.392360	3.75
3	1.15	1.15	1.583424	6.282235	1.583424	1.683500	5.94
4	1.2	1.2	1.897536	8.197355	1.897536	2.070935	8.37
5	1.25	1.25	2.307404	10.815955	2.307404	2.593803	11.04
6	1.3	1.3	2.848201	14.440381	2.848201	3.310171	13.96
7	1.35	1.35	3.570220	19.520180	3.570220	4.307575	17.12
8	1.4	1.4	4.546229	26.731829	4.546229	5.720178	20.52
9	1.45	1.45	5.882821	37.105893	5.882821	7.757228	24.16
10	1.5	1.5	7.738116	52.232280	7.738116	10.751013	28.02
11	1.55	1.55	10.349730	74.595676	10.349730	15.239260	32.09
12	1.6	1.6	14.079513	108.130663	14.079513	22.109337	36.32
13	1.65	1.65	19.486047	159.152285	19.486047	32.855692	40.69
14	1.7	1.7	27.443661	237.936540	27.443661	50.048874	45.17
15	1.75	1.75	39.340488	361.440732	39.340488	78.208239	49.70
16	1.8	1.8	57.412524	558.049737	57.412524	125.461633	54.24
17	1.85	1.85	85.315011	875.971879	85.315011	206.773709	58.74
18	1.9	1.9	129.113605	1398.300345	129.113605	350.373595	63.15
19	1.95	1.95	199.028622	2270.419011	199.028622	610.864398	67.42
20	2	2	312.549573	3750.594876	312.549573	1096.633158	71.50

Taylor Series Expansion

■ Example 7 – Euler's Method

Repeat Example 5 for $1 \le x \ge 2$ with a step size of *h* = 0.02 :

$$\frac{dy}{dx} = 3x^2 y \qquad \text{such that } y = 1 \text{ at } x = 1$$

$x_0 = 1$

$y_0 = 1$

Again we have

$$y(1) = 1 \qquad \text{or} \qquad x_0 = 1$$

$$y_0 = 1$$

First-order Ordinary Differential Equations

- Example 7 (cont'd) – Euler's Method

First Iteration ($i = 0$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$x_0 = 1, \quad y_0 = 1, \text{ and } \quad h = 0.02$$
$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{\substack{x_0=1 \\ y_0=1}} = 3x^2 y = 3(1)^2(1) = 3$$
$$\therefore y_1 = 1 + 0.02(3) = 1 + 0.06 = 1.06$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 7 (cont'd) – Euler's Method

Second Iteration ($i = 1$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$x_1 = 1.02, \quad y_1 = 1.06, \text{ and } \quad h = 0.02$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=1.02 \\ y_0=1.06}} = 3x^2 y = 3(1.02)^2(1.06) = 3.30847$$
$$\therefore y_2 = 1.06 + 0.02(3.30847) = 1.12617$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 7 (cont'd) – Euler's Method

Third Iteration ($i = 2$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$x_2 = 1.04, \quad y_2 = 1.12617, \text{ and } \quad h = 0.02$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=1.04 \\ y_0=1.12617}} = 3x^2 y = 3(1.04)^2(1.12617) = 3.65420$$
$$\therefore y_3 = 1.12617 + 0.02(3.65420) = 1.19925$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 7 (cont'd) – Euler's Method

– See the spreadsheet output in the next viewgraph for the rest of the iterations.

– From Example 5, the expression for the exact solution is

$$y = e^{x^3 - 1}$$

First-order Ordinary Differential Equations

■ Example 7 (cont'd) – Euler's Method

True Function :

$$y = e^{x^3 - 1}$$

$$h = 0.02$$

i	x	x_i	y_i	$f(x_i, y_i)$	y (Euler)	y (True)	% Error
0	1	1	1	3		1	
1	1.02	1.02	1.060000	3.308472	1.060000	1.063120	0.29
2	1.04	1.04	1.126169	3.654195	1.126169	1.132994	0.60
3	1.06	1.06	1.199253	4.042443	1.199253	1.210479	0.93
4	1.08	1.08	1.280102	4.479334	1.280102	1.296557	1.27
5	1.1	1.1	1.369689	4.971971	1.369689	1.392360	1.63
6	1.12	1.12	1.469128	5.528624	1.469128	1.499195	2.01
7	1.14	1.14	1.579701	6.158937	1.579701	1.618572	2.40
8	1.16	1.16	1.702879	6.874184	1.702879	1.752242	2.82
9	1.18	1.18	1.840363	7.687565	1.840363	1.902240	3.25
10	1.2	1.2	1.994114	8.614575	1.994114	2.070935	3.71
11	1.22	1.22	2.166406	9.673436	2.166406	2.261092	4.19
12	1.24	1.24	2.359875	10.8856299	2.359875	2.475950	4.69
13	1.26	1.26	2.577587	12.2765327	2.577587	2.719304	5.21
14	1.28	1.28	2.823118	13.8761893	2.823118	2.995622	5.76
15	1.3	1.3	3.100642	15.7202535	3.100642	3.310171	6.33
16	1.32	1.32	3.415047	17.8511326	3.415047	3.669179	6.93
17	1.34	1.34	3.772069	20.3193837	3.772069	4.080029	7.55
18	1.36	1.36	4.178457	23.1854229	4.178457	4.551496	8.20
19	1.38	1.38	4.642166	26.5216204	4.642166	5.094044	8.87
20	1.4	1.4	5.172598	30.4148761	5.172598	5.720178	9.57

First-order Ordinary Differential Equations

■ Example 7 (cont'd) – Euler's Method

True Function :

$$y = e^{x^3 - 1}$$

$$h = 0.02$$

i	x	x_i	y_i	$f(x_i, y_i)$	y (Euler)	y (True)	% Error
21	1.42	1.42	5.780896	34.96979	5.780896	6.444893	10.30269
22	1.44	1.44	6.480291	40.3126	6.480291	7.286213	11.06092
23	1.46	1.46	7.286543	46.59599	7.286543	8.265878	11.84792
24	1.48	1.48	8.218463	54.00516	8.218463	9.410179	12.66412
25	1.5	1.5	9.298566	62.76532	9.298566	10.75101	13.50986
26	1.52	1.52	10.55387	73.151	10.55387	12.3272	14.38547
27	1.54	1.54	12.01689	85.49779	12.01689	14.18612	15.29119
28	1.56	1.56	13.72685	100.217	13.72685	16.38581	16.22724
29	1.58	1.58	15.73119	117.814	15.73119	18.99759	17.19376
30	1.6	1.6	18.08747	138.9118	18.08747	22.10934	18.19081
31	1.62	1.62	20.8657	164.2799	20.8657	25.82978	19.21842
32	1.64	1.64	24.1513	194.872	24.1513	30.29383	20.2765
33	1.66	1.66	28.04874	231.8733	28.04874	35.6695	21.36492
34	1.68	1.68	32.68621	276.7607	32.68621	42.16675	22.48346
35	1.7	1.7	38.22142	331.3797	38.22142	50.04887	23.63181
36	1.72	1.72	44.84902	398.044	44.84902	59.64725	24.80958
37	1.74	1.74	52.80989	479.6617	52.80989	71.38045	26.0163
38	1.76	1.76	62.40313	579.8998	62.40313	85.77915	27.2514
39	1.78	1.78	74.00113	703.3955	74.00113	103.5187	28.51422
40	1.8	1.8	88.06903	856.031	88.06903	125.4616	29.80401

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Errors with Euler's Method
  - Examining Examples 5 through 7, we find that different degrees of errors can be generated depending on the the step size with Euler's method.
  - In general, as the step size is made smaller, the errors are reduced and the accuracy of the solution improves.

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Errors with Euler's Method
  - Errors can be classified as
    - Global
    - Local
  - The *global* errors are cumulative over the range of solution.
  - The *local* errors occur over one step size.

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Errors with Euler's Method

– The global errors can be computed as the difference between the numerical and exact solution.

– The local errors can be computed as the difference between a numerical solution at the end of the step using the exact value at the beginning of the step and the exact solution at the end of the step.

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Errors with Euler's Method

– The error, ε, using Euler's method can be approximated using the third term of the Taylor series expansion as

$$\varepsilon = \frac{(x - x_0)^2}{2!}\frac{d^2 y}{dx^2} \qquad (14)$$

• The second derivative is evaluated at the point where it is maximum over the interval $(x - x_0)$.

• If the range is divided into n increments, then the error at the end of the range of interest for x would be $n\ \varepsilon$.

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 - Errors with Euler's Method

Analyze both the approximate and actual errors of Example 5, 6, and 7 when x is equal to 1.1 for step sizes of 0.1, 0.05, and 0.02.

The approximate error is given by Eq. 14 as

$$\varepsilon = \frac{(x - x_0)^2}{2!}\frac{d^2 y}{dx^2}$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

For $h = 0.1$

$$\frac{dy}{dx} = 3x^2 y$$

$$\frac{d^2 y}{dx^2} = 6xy + 3x^2 \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = 6xy + 3x^2(3x^2 y) = 6xy + 9x^4 y$$

$$\left.\frac{d^2 y}{dx^2}\right|_{\substack{x=1.1 \\ y=1.3}} = 6(1.1)(1.3) + 9(1.1)^4(1.3) = 25.70997$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

For $h = 0.05$

$$\frac{dy}{dx} = 3x^2 y$$

$$\frac{d^2 y}{dx^2} = 6xy + 3x^2 \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = 6xy + 3x^2(3x^2 y) = 6xy + 9x^4 y$$

$$\left.\frac{d^2 y}{dx^2}\right|_{\substack{x=1.1 \\ y=1.34018}} = 6(1.1)(1.34018) + 9(1.1)^4(1.34018) = 26.50461$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

For $h = 0.02$

$$\frac{dy}{dx} = 3x^2 y$$

$$\frac{d^2 y}{dx^2} = 6xy + 3x^2 \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = 6xy + 3x^2(3x^2 y) = 6xy + 9x^4 y$$

$$\left.\frac{d^2 y}{dx^2}\right|_{\substack{x=1.1 \\ y=1.369689}} = 6(1.1)(1.369689) + 9(1.1)^4(1.369689) = 27.088202$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

$$\text{Error} = n\varepsilon = n\frac{h^2}{2}\frac{d^2y}{dx^2}\bigg|_{\substack{x=1.1\\ y=1.3,\,1.34018,\,1.369689}}$$

$$\text{For } h = 0.1 \Rightarrow \text{approx. error} = (1)\frac{(0.1)^2}{2}(25.70997) = \underline{0.12855}$$

$$\text{For } h = 0.05 \Rightarrow \text{approx. error} = (2)\frac{(0.05)^2}{2}(26.50461) = \underline{0.066262}$$

$$\text{For } h = 0.02 \Rightarrow \text{approx. error} = (5)\frac{(0.02)^2}{2}(27.088202) = \underline{0.027088}$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

$h = 0.1$

x	Exact Solution	Numerical Solution	Local Error	Global Error	Local Error (%)	Global Error (%)
1	1	1	0	0	0	0
1.1	1.392360	1.300000	0.092360	0.092360	6.633	6.633
1.2	2.070935	1.771900	0.206675	0.299035	9.980	14.440
1.3	3.310171	2.537361	0.473776	0.772811	14.313	23.347
1.4	5.720178	3.823803	1.123565	1.896376	19.642	33.152
1.5	10.751013	6.072199	2.782439	4.678814	25.881	43.520
1.6	22.109337	10.170933	7.259589	11.938404	32.835	53.997
1.7	50.048874	17.982209	20.128260	32.066664	40.217	64.071
1.8	125.461633	33.572785	59.822184	91.888848	47.682	73.241
1.9	350.373595	66.205532	192.279215	284.168064	54.878	81.104
2	1096.633158	137.906122	674.558973	958.727036	61.512	87.425

$$\text{Approximate Error} = 0.12855$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

$h = 0.05$

Approximate Error = 0.066262

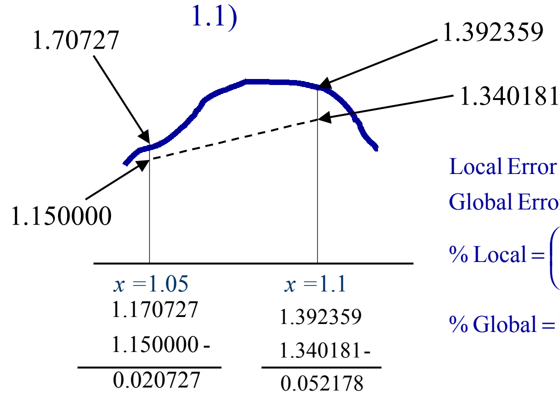
x	Exact Solution	Numerical Solution	Local Error	Global Error	Local Error (%)	Global Error (%)
1.00	1	1	0	0	0	0
1.05	1.17072709	1.15	0.020727	0.020727	1.770	1.770
1.10	1.39235979	1.340181	0.031451	0.052179	2.259	3.747
1.15	1.68350007	1.583424	0.047897	0.100076	2.845	5.945
1.20	2.07093459	1.897536	0.073323	0.173399	3.541	8.373
1.25	2.59380264	2.307404	0.113000	0.286399	4.357	11.042
1.30	3.3101715	2.848201	0.175571	0.461970	5.304	13.956
1.35	4.30757457	3.57022	0.275384	0.737354	6.393	17.118
1.40	5.72017844	4.546229	0.436595	1.173949	7.633	20.523
1.45	7.75722758	5.882821	0.700458	1.874407	9.030	24.163
1.50	10.7510132	7.738116	1.138491	3.012898	10.590	28.024
1.55	15.2392601	10.34973	1.876633	4.889530	12.314	32.085
1.60	22.1093368	14.07951	3.140293	8.029823	14.203	36.319
1.65	32.8556919	19.48605	5.339822	13.369645	16.252	40.692
1.70	50.0488736	27.44366	9.235567	22.605213	18.453	45.166
1.75	78.208239	39.34049	16.262538	38.867751	20.794	49.698
1.80	125.461633	57.41252	29.181358	68.049109	23.259	54.239
1.85	206.773709	85.31501	53.409589	121.458697	25.830	58.740
1.90	350.373595	129.1136	99.801293	221.259990	28.484	63.150
1.95	610.864398	199.0286	190.575786	411.835776	31.198	67.419
2.00	1096.63316	312.5496	372.247810	784.083585	33.945	71.499

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

– Sample Calculations for Actual Errors ($h = 0.05$, $x = 1.1$)



$$\text{Local Error} = 0.052178 - 0.020727 = 0.031451$$

$$\text{Global Error} = 1.392359 - 1.340181 = 0.052178$$

$$\%\ \text{Local} = \left(\frac{0.031451}{1.392359}\right)\times 100 = 2.259$$

$$\%\ \text{Global} = \left(\frac{1.392359 - 1.340181}{1.392359}\right)\times 100 = 3.747$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 8 (cont'd) - Errors with Euler's Method

$h = 0.02$

Approximate Error = 0.027088

x	Exact Solution	Numerical Solution	Local Error	Global Error	Local Error (%)	Global Error (%)
1	1	1	0	0	0	0
1.02	1.06312	1.06	0.003120	0.003120	0.293	0.293
1.04	1.132994	1.126169	0.003705	0.006825	0.327	0.602
1.06	1.210479	1.199253	0.004401	0.011225	0.364	0.927
1.08	1.296557	1.280102	0.005229	0.016454	0.403	1.269
1.1	1.39236	1.369689	0.006216	0.022671	0.446	1.628
1.12	1.499195	1.469128	0.007395	0.030066	0.493	2.005
1.14	1.618572	1.579701	0.008805	0.038871	0.544	2.402
1.16	1.752242	1.702879	0.010492	0.049362	0.599	2.817
1.18	1.90224	1.840363	0.012514	0.061877	0.658	3.253
1.2	2.070935	1.994114	0.014944	0.076820	0.722	3.709
1.22	2.261092	2.166406	0.017866	0.094686	0.790	4.188
1.24	2.47595	2.359875	0.021389	0.116075	0.864	4.688
1.26	2.719304	2.577587	0.025642	0.141717	0.943	5.212
1.28	2.995622	2.823118	0.030788	0.172504	1.028	5.759
1.3	3.310171	3.100642	0.037025	0.209530	1.119	6.330
1.32	3.669179	3.415047	0.044603	0.254132	1.216	6.926
1.34	4.080029	3.772069	0.053827	0.307959	1.319	7.548
1.36	4.551496	4.178457	0.065080	0.373039	1.430	8.196
1.38	5.094044	4.642166	0.078839	0.451878	1.548	8.871
1.4	5.720178	5.172598	0.095702	0.547580	1.673	9.573

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Modified Euler's Method

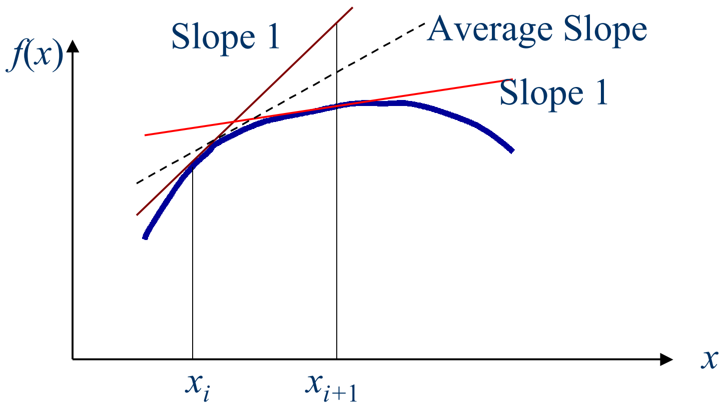
– The modified Euler's method attempts to improve the accuracy of the estimated solution by using only the first derivative as in the basic Euler's method, but it uses an average slope, rather than the slope at the start of the interval (step).

– It is structurally similar to the basic Euler's method.

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Modified Euler's Method



First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Modified Euler's Method

General Steps:

1. Evaluate the slope dy/dx at the start of the interval.
2. Using Euler's basic method, estimate y at the end of the interval at x_{i+1}.
3. Evaluate the slope dy/dx at the end of the interval at x_{i+1}.
4. Find the average slope as given by the following equation:

First-order Ordinary Differential Equations

■ Modified Euler's Method

$$\left.\frac{dy}{dx}\right|_{\text{average}} = \frac{\left.\frac{dy}{dx}\right|_{x=x_i} + \left.\frac{dy}{dx}\right|_{x=x_{i+1}}}{2} \qquad (15)$$

5. Compute the revised y at the end of the interval using

$$y_{i+1} = y_i + h\left.\frac{dy}{dx}\right|_{\text{average}} \qquad (16)$$

First-order Ordinary Differential Equations

■ Modified Euler's Method

– Summary

• Modified Euler's method can be given by the following equivalent expressions:

$$y_{i+1} = y_i + h\left.\frac{dy}{dx}\right|_{\text{average}} \qquad (17)$$

or

$$y_{i+1} = y_i + h\frac{\left[f(x_i, y_i) + f\left(x_i + h, y_i + hf(x_i, y_i)\right)\right]}{2} \qquad (18)$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Modified Euler's Method

$$y_{i+1} = y_i + 0.5h\left[\underbrace{f(x_i, y_i)}_{\left.\frac{dy}{dx}\right|_{x_i}} + \underbrace{f\left(x_i + h, y_i + hf(x_i, y_i)\right)}_{\left.\frac{dy}{dx}\right|_{x_{i+1}}}\right]$$

Slope 1 S_1

Slope 2 S_2

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

■ Example 9 – Modified Euler's Method

Solve the following differential equation using the modified Euler's method for $0 \leq x \geq 1$ with step size of $h = 0$:

$$\frac{dy}{dx} - \frac{1}{2}y = 0 \qquad \text{such that } y = 1 \text{ at } x = 0$$

First-order Ordinary Differential Equations

- Example 9 (cont'd) – Modified Euler's Method

$$i = 0, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.1$$

Step 1:

$$\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} = \frac{1}{2}y = \frac{1}{2}(1) = \frac{1}{2}$$

Step 1:

$$y_1^{Euler} = y_0 + h\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} = 1 + 0.1\frac{1}{2} = 1.05$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 9 (cont'd) – Modified Euler's Method

Step 3:

$$\left.\frac{dy}{dx}\right|_{\substack{x=0.1\\y=1.05}} = \frac{1}{2}y = \frac{1}{2}(1.05) = 0.525$$

Step 4:

$$\left.\frac{dy}{dx}\right|_{\text{average}} = \frac{0.5 + 0.525}{2} = 0.5125$$

Step 5:

$$y_1 = y_0 + h\left.\frac{dy}{dx}\right|_{\text{average}} = 1 + 0.1(0.5125) = 1.05125$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 9 (cont'd) – Modified Euler's Method

$$i = 1, \quad x_0 = 0.1, \quad y_0 = 1.05125, \quad h = 0.1$$

Step 1:

$$\left.\frac{dy}{dx}\right|_{\substack{x=0.1 \\ y=1.05125}} = \frac{1}{2}y = \frac{1}{2}(1.05125) = 0.52563$$

Step 1:

$$y_2^{Euler} = y_1 + h\left.\frac{dy}{dx}\right|_{\substack{x=0.1 \\ y=1.05125}} = 1.05125 + (0.1)(0.52563) = 1.10381$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 9 (cont'd) – Modified Euler's Method

Step 3:

$$\left.\frac{dy}{dx}\right|_{\substack{x=0.2 \\ y=1.10381}} = \frac{1}{2}y = \frac{1}{2}(1.10381) = 0.55191$$

Step 4:

$$\left.\frac{dy}{dx}\right|_{\text{average}} = \frac{0.52563 + 0.55191}{2} = 0.53877$$

Step 5:

$$y_2 = y_1 + h\left.\frac{dy}{dx}\right|_{\text{average}} = 1.05125 + 0.1(0.53877) = 1.10513$$

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 9 (cont'd) – Modified Euler's Method $\frac{dy}{dx} - \frac{1}{2}y = 0$ such that $y = 1$ at $x = 0$

Exact Solution is given by

$$y = e^{\frac{x}{2}}$$

i	x	$f(x_i, y_i)$	$f(x_{i+1}, y_{i+1})$	y (modified Euler)	y (exact)	% error
0	0	0.500000	-	-	1	-
1	0.1	0.525625	0.525	1.05125	1.051271	0.00
2	0.2	0.552563	0.551906	1.105126563	1.105171	0.00
3	0.3	0.580882	0.580191	1.161764299	1.161834	0.01
4	0.4	0.610652	0.609926	1.221304719	1.221403	0.01
5	0.5	0.641948	0.641185	1.283896586	1.284025	0.01
6	0.6	0.674848	0.674046	1.349696286	1.349859	0.01
7	0.7	0.709434	0.708591	1.418868221	1.419068	0.01
8	0.8	0.745793	0.744906	1.491585217	1.491825	0.02
9	0.9	0.784014	0.783082	1.568028959	1.568312	0.02
10	1	0.824195	0.823215	1.648390444	1.648721	0.02

First-order Ordinary Differential Equations

• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering

- Example 9 (cont'd) – Modified Euler's Method $\frac{dy}{dx} - \frac{1}{2}y = 0$ such that $y = 1$ at $x = 0$

i	x	$f(x_i, y_i)$	$f(x_{i+1}, y_{i+1})$	y (Modified Euler)	y (Basic Euler Example 3)	% error
0	0	0.500000	-	-		-
1	0.1	0.525625	0.525	1.05125	1.05	0.12
2	0.2	0.552563	0.551906	1.10512656	1.1025	0.24
3	0.3	0.580882	0.580191	1.1617643	1.157625	0.36
4	0.4	0.610652	0.609926	1.22130472	1.21550625	0.47
5	0.5	0.641948	0.641185	1.28389659	1.276281563	0.59
6	0.6	0.674848	0.674046	1.34969629	1.340095641	0.71
7	0.7	0.709434	0.708591	1.41886822	1.407100423	0.83
8	0.8	0.745793	0.744906	1.49158522	1.477455444	0.95
9	0.9	0.784014	0.783082	1.56802896	1.551328216	1.07
10	1	0.824195	0.823215	1.64839044	1.628894627	1.18