CHAPTER 8. DIFFERENTIAL EQUATIONS

Introduction

- Differential equations are used extensively in engineering and science to represent physical phenomena of a problem (problems).
- A differential equation is any equation containing one or more derivative terms.
- An ordinary differential equation is that involves a single independent variable.
Introduction

- Differential equations involving two or more independent variables are referred to as partial differential equations.
- The analytical solutions of both ordinary and partial differential equations is called "closed-form solution".
- This solution requires the constant of integration be evaluated by using prescribed values of the independent variable(s).

Introduction

- Classification of Differential Equations
  - Ordinary Differential Equations
    - First-order
    - Higher-order
    - Linear
    - Nonlinear
  - Partial Differential Equations
    - These equations are usually classified according to their mathematical form.
**Introduction**

■ Ordinary Differential Equations

– The general forms of an ordinary differential equation is given by one of the following expressions:

\[ C_0(x) + \sum_{i=1}^{n} C_i(x) \frac{d^i}{dx^i} = 0 \]  \hspace{1cm} (1)

\[ C_0(x) + \sum_{i=1}^{n} C_i(x) \left( \frac{d^i}{dx^i} \right)^m = 0 \hspace{1cm} (m \neq 0) \]  \hspace{1cm} (2)

– Note that Eq. 1 is a linear ordinary differential equation, while Eq. 2 is a nonlinear differential equation.

– Furthermore, if the coefficient \( C_0(x) \) is zero, the equation is called homogenous, otherwise nonhomogenous.

– ODE’s of all types have many applications in engineering and science.
Introduction

- Examples Ordinary Differential Equations

\[
\frac{dy}{dx} = 5x \\
\frac{dy}{dx} - \frac{1}{2} = 0 \\
\frac{d^2y}{dx^2} - x + y = 0
\]

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- Partial Differential Equations
  - Differential equations involving two or more independent variables are called partial differential equations.
  - These equations may have only boundary conditions, in which they are referred to as *Boundary Value Problems (BVP)* or steady-state equations.
Introduction

- Partial Differential Equations
  - In some applications both boundary and initial conditions are specified.
  - In these cases, they are called \textit{transient problems}.
  - In practice, very few partial differential equations have closed-form analytical solution. Therefore, numerical techniques are required in most cases.

Introduction

- Examples Partial Differential Equations
  \[
  \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
  \]
  \[
  C_1 \frac{\partial^2 w}{\partial x^2} + C_2 \frac{\partial^2 w}{\partial x \partial y} + C_3 \frac{\partial^2 w}{\partial y^2} + C_4 = 0
  \]
  \[
  \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{F_z}{D} - \frac{k}{D} w
  \]
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- Engineering Examples
  - Mechanical System

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F(t) \]

- Electrical Circuit

\[ L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dv}{dt} \]
Introduction

- **Engineering Examples**
  - **Vibrating Beam**

  \[ m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = F(t) \]

- **Steady-state Fluid Flow under Dams**

  \[ k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} = 0 \]
Introduction

- Engineering Examples
  - Plate on Elastic Foundation

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{F_z}{D} - \frac{k}{D} w \]

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- Engineering Examples
  - Transient Temperature Distribution

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = C \frac{\partial T}{\partial t} \]
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- General Expressions for Differential Equations

\[
\begin{align*}
\frac{dy}{dx} &= f(x, y) \\
\frac{dy}{dx} &= f(x) \\
\frac{dy}{dx} &= f(y) \\
\frac{dy}{dx} &= C \\
\frac{dy}{dx} &= 2 + x^2 - y \\
\frac{dy}{dx} &= 3 - \frac{1}{x} \\
\frac{dy}{dx} &= e^x - 2 \\
\frac{dy}{dx} &= -4 \\
\end{align*}
\]
**Introduction**

### General Expressions for Differential Equations

\[
\begin{align*}
\frac{dy}{dx} &= f(x_1, x_2, y) \\
\frac{dy}{dx} &= f(x_1, x_2) \\
\frac{dy}{dx} &= f(x_1, y)
\end{align*}
\]

\[
\begin{align*}
\frac{dy}{dx} &= x_1^2 - x_2^{-1} - 2y^2 \\
\frac{dy}{dx} &= \frac{1}{x_1} - x_2^2 \\
\frac{dy}{dx} &= f(x_1, y) = 3x_1 + 3y^2 - 1
\end{align*}
\]

### Origin of Differential Equations

- They can originate from either geometric or physical problems.
- For geometric case, consider the slope of a function, which is usually a relationship between \(y\) and \(x\).
- This slope illustrates the geometric case, and is given by

\[
\frac{dy}{dx} = c(y - x)
\]

\[\text{(3)}\]
Introduction

■ Origin of Differential Equations
  – The solution of Eq. 3 would a relationship of the form
    \[ y = g(x) \]  \hspace{1cm} \text{(4)}
  – Equation 4 may be subject to one or more boundary conditions.
  – Physical problems can also be defined by differential equations.

■ Origin of Differential Equations
  – As we saw earlier, simple problems in electrical circuits and heat transfer involve differential equations.
  – Simple problems motions can also be expressed by differential equations. For example the gravitational equation is
    \[ F = ma = m \frac{dV}{dt} \]