CHAPTER 6b. NUMERICAL INTERPOLATION

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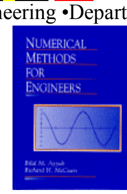

by

Dr. Ibrahim A. Assakkaf

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering
University of Maryland, College Park

Method of Undetermined Coefficients

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- Example 6
  - Develop a fourth-order interpolation polynomial for the following set of data, for which we know their original function, that is, $f(x) = x^3$.

x	0	1	2	3	4
$f(x)$	0	1	8	27	64

Method of Undetermined Coefficients

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- Example 6 (cont'd)
  - The general form of the interpolation polynomial is given by Eq. 4 as

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n$$

  - In our case it is

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$$

Method of Undetermined Coefficients

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- Example 6 (cont'd)
  - We need to find the coefficients b_0, b_1, b_2, b_3 and b_4.
  - We notice that have five unknowns,$(n + 1) = 4 + 1 = 4$, that require solving 5 simultaneous linear equations using the pair, x_i and $f(x_i)$, of the given data as follows:

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- Example 6 (cont'd)

x	0	1	2	3	4
$f(x)$	0	1	8	27	64

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$$

$$0 = b_0 + b_1(0) + b_2(0)^2 + b_3(0)^3 + b_4(0)^4$$
$$1 = b_0 + b_1(1) + b_2(1)^2 + b_3(1)^3 + b_4(1)^4$$
$$8 = b_0 + b_1(2) + b_2(2)^2 + b_3(2)^3 + b_4(2)^4$$
$$27 = b_0 + b_1(3) + b_2(3)^2 + b_3(3)^3 + b_4(3)^4$$
$$64 = b_0 + b_1(4) + b_2(4)^2 + b_3(4)^3 + b_4(4)^4$$

Method of Undetermined Coefficients

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- Example 6 (cont'd)

$$
\begin{aligned}
b_0 &= 0 \\
b_0 + b_1 + b_2 + b_3 + b_4 &= 1 \\
b_0 + 2b_1 + 4b_2 + 8b_3 + 16b_4 &= 8 \\
b_0 + 3b_1 + 9b_2 + 27b_3 + 81b_4 &= 27 \\
b_0 + 4b_1 + 16b_2 + 64b_3 + 256b_4 &= 64
\end{aligned}
$$

– OR

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \\ 64 \end{bmatrix}$$

Method of Undetermined Coefficients

- Example 6 (cont'd)
  - The solution of this set of equations yields

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$$

  - Therefore, the interpolation polynomial is

$$f(x) = x^3$$

Method of Undetermined Coefficients

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- Example 6 (cont'd)
  - We notice in this example that as the degree of the interpolation polynomial goes higher, the computational effort becomes complex.
  - When the polynomial is of the same order as the function used to derive that data, the polynomial will provide errorless interpolated values, and will be exactly the same as the original function.

Gregory-Newton Interpolation Method

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- The previous example shows that the fifth-order interpolation polynomial gives errorless interpolated values, and there was significant improvement in accuracy.
- Thus, it is usually tempting to try an interpolation polynomial of higher order to improve the accuracy.

Gregory-Newton Interpolation Method

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- However, it should be recognized that the solution procedure becomes more complex as the order increases.
- The solution in this case would require more simultaneous equations to be solved for the coefficients.

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Gregory-Newton Interpolation Method

- The *Gregory-Newton Method* provides a means of developing an nth-order interpolation polynomial without requiring the solution of a set of simultaneous linear equations.

Gregory-Newton Interpolation Method

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- Gregory-Newton Formula

$$\begin{aligned} f(x) = a_1 &+ a_2(x-x_1) + a_3(x-x_1)(x-x_2) + a_4(x-x_1)(x-x_2)(x-x_3) \\ &+ a_5(x-x_1)(x-x_2)(x-x_3)(x-x_4) + \cdots \\ &+ a_n(x-x_1)(x-x_2)\ldots(x-x_{n-1}) + a_{n+1}(x-x_1)(x-x_2)\ldots(x-x_n) \end{aligned} \tag{7}$$

Where

x = independent variable

x_i = n known values for i = 1, 2, …, n

a_i = n + 1 unknown coefficients for i = 1, 2, …, (n + 1)

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■ Gregory-Newton Formula

– The $(n + 1)$ coefficients can developed from a set of $(n + 1)$ values of the depended variables as given in tabular form as

i	1	2	3	…	n
x	x_1	x_2	x_3	…	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$

Gregory-Newton Interpolation Method

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■ Gregory-Newton Formula

– If $x = x_1$, then $f(x) = f(x_1)$

for this pair of values, every term in Eq. 7 except a_1 is zero, that is

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + a_4(x - x_1)(x - x_2)(x - x_3) + a_5(x - x_1)(x - x_2)(x - x_3)(x - x_4) + \cdots$$

$$f(x_1) = a_1 + a_2\underbrace{(x_1 - x_1)}_{0} + a_3\underbrace{(x_1 - x_1)}_{0}(x_1 - x_2) + a_4\underbrace{(x_1 - x_1)}_{0}(x_1 - x_2)(x_1 - x_3) + a_5\underbrace{(x_1 - x_1)}_{0}(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) + \cdots$$

Gregory-Newton Interpolation Method

■ Gregory-Newton Formula

– Therefore,

$$a_1 = f(x_1) \qquad (8)$$

Gregory-Newton Interpolation Method

■ Gregory-Newton Formula

– If $x = x_2$, then $f(x) = f(x_2)$

for this pair of values, every term in Eq. 7 except a_1 and a_2 are zero, that is

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + a_4(x - x_1)(x - x_2)(x - x_3)$$
$$+ a_5(x - x_1)(x - x_2)(x - x_3)(x - x_4) + \cdots$$

$$f(x_2) = a_1 + a_2(x_2 - x_1) + a_3(x_2 - x_1)\overbrace{(x_2 - x_2)}^{0} + a_4(x_2 - x_1)\overbrace{(x_2 - x_2)}^{0}(x_2 - x_3)$$
$$+ a_5(x_2 - x_1)\overbrace{(x_2 - x_2)}^{0}(x_2 - x_3)(x_2 - x_4) + \cdots$$

Gregory-Newton Interpolation Method

■ Gregory-Newton Formula

– Therefore,

$$f(x_2) = a_1 + a_2(x_2 - x_1)$$
$$= f(x_1) + a_2(x_2 - x_1)$$

OR

$$a_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad (9)$$

Gregory-Newton Interpolation Method

■ Gregory-Newton Formula

– If $x = x_3$, then $f(x) = f(x_3)$

for this pair of values, every term in Eq. 7 except a_1, a_2 and a_3 are zero, that is

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + a_4(x - x_1)(x - x_2)(x - x_3)$$
$$+ a_5(x - x_1)(x - x_2)(x - x_3)(x - x_4) + \cdots$$

$$f(x_3) = a_1 + a_2(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2) + a_4(x_3 - x_1)(x_3 - x_2)\underbrace{(x_3 - x_3)}_{0}$$
$$+ a_5(x_3 - x_1)(x_3 - x_2)\underbrace{(x_3 - x_3)}_{0}(x_3 - x_4) + \cdots$$

Gregory-Newton Interpolation Method

■ Gregory-Newton Formula

– Therefore,

$$f(x_3) = a_1 + a_2(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2)$$

$$f(x_3) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2)$$

OR

$$a_3 = \frac{f(x_3) - f(x_1) - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_1)}{(x_3 - x_1)(x_3 - x_2)} \qquad (10)$$

Gregory-Newton Interpolation Method

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■ Gregory-Newton Formula

- The process can be repeated to solve the remaining coefficients.
- At each step a new pair of points is used with the values of a_i computed in previous step.
- Note that the Gregory-Newton method yields the same polynomial as the solution of the simultaneous equations used for the method of undetermined coefficients.

Gregory-Newton Interpolation Method

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- Gregory-Newton Formula
  - Thus, it will give the same interpolated value and have the same accuracy,
  - The advantage of the Gregory-Newton Method is that it does not require the solution of n simultaneous equations with n unknowns.

Gregory-Newton Interpolation Method

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- Example 7: G-N Method

  Fit a quadratic polynomial for the following set of data using the Gregory-Newton interpolation method. Using this polynomial approximate the function $f(x)$ when x is equal to 2.2 and 2.8.

x	1	2	3
$f(x)$	1	8	27

Gregory-Newton Interpolation Method

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- Example 7 (cont'd): G-N Method

If $x = 1$, then $f(x) = 1$

$$1 = a_1 + a_2(1-1) \Rightarrow a_1 = 1$$

If $x = 2$, then $f(x) = 8$

$$8 = a_1 + a_2(2-1) + a_3(2-1)(2-2)$$

$$8 = 1 + a_2 \Rightarrow a_2 = 7$$

Gregory-Newton Interpolation Method

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- Example 7 (cont'd): G-N Method

If $x = 3$, then $f(x) = 27$

$$27 = a_1 + a_2(3-1) + a_3(3-1)(3-2) + a_4(3-1)(3-2)(3-3)$$

$$27 = 1 + 7(2) + 2a_3 \Rightarrow a_3 = 6$$

Hence, the polynomial is given by

$$f(x) = 1 + 7(x-1) + 6(x-1)(x-2)$$

Gregory-Newton Interpolation Method

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- Example 7 (cont'd): G-N Method

For $x = 2.2$,

$$f(x) = 1 + 7(x-1) + 6(x-1)(x-2)$$
$$f(2.2) = 1 + 7(2.2-1) + 6(2.2-1)(2.2-2) = 10.84$$

For $x = 2.8$

$$f(x) = 1 + 7(x-1) + 6(x-1)(x-2)$$
$$f(2.8) = 1 + 7(2.8-1) + 6(2.8-1)(2.8-2) = 22.24$$

Gregory-Newton Interpolation Method

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- Example 7 (cont'd): G-N Method
  - Comparison With true values:

x	$f(x)$		%Error
	Approximation	True	
2.2	10.84	10.65	1.78
2.8	22.24	21.95	1.32

Gregory-Newton Interpolation Method

- Example 8: G-N Interpolation

  Show that the quadratic polynomial obtained in Example 7 could have been obtained using the method of undetermined coefficients that will result in the same interpolated values for x equal 2.2 and 2.8.

Gregory-Newton Interpolation Method

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- Example 8 (cont'd): G-N Interpolation
  - A quadratic polynomial using the method of undetermined coefficients is given by

$$f(x) = b_0 + b_1 x + b_2 x^2$$

  - The coefficients will be determined as follows

x	1	2	3
$f(x)$	1	8	27

Gregory-Newton Interpolation Method

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- Example 8 (cont'd): G-N Interpolation

x	1	2	3
$f(x)$	1	8	27

$$1 = b_0 + b_1(1) + b_2(1)^2$$
$$8 = b_0 + b_1(2) + b_2(2)^2$$
$$27 = b_0 + b_1(3) + b_2(3)^2$$

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- Example 8 (cont'd): G-N Interpolation

$$1 = b_0 + b_1(1) + b_2(1)^2 \qquad\qquad b_0 + b_1 + b_2 = 1$$
$$8 = b_0 + b_1(2) + b_2(2)^2 \quad \Longrightarrow \quad b_0 + 2b_1 + 4b_2 = 8$$
$$27 = b_0 + b_1(3) + b_2(3)^2 \qquad\qquad b_0 + 3b_1 + 9b_2 = 27$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 27 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 6 \end{bmatrix}$$

Gregory-Newton Interpolation Method

- Example 8 (cont'd): G-N Interpolation
  - Therefore,

$$f(x) = b_0 + b_1 x + b_2 x^2$$

OR

$$f(x) = 6 - 11x + 6x^2$$

$$f(2.2) = 6 - 11(2.2) + 6(2.2)^2 = 10.84$$
$$f(2.8) = 6 - 11(2.8) + 6(2.8)^2 = 22.24$$

These are the same values obtained by Gregory-Newton Method.