CHAPTER 5f. SIMULTANEOUS LINEAR EQUATIONS

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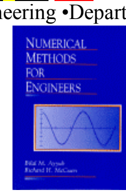

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Banded Matrices

- A banded matrix was defined earlier as square matrix with elements of zero except on the principal diagonal and the values in the positions adjacent to the principal diagonal.
- A tridiagonal matrix is a special case of a banded matrix.

Banded Matrices

■ Tridiagonal Matrix

– A triadigonal matrix is a special case of a banded matrix that has zeros except in the three diagonals:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{32} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

Banded Matrices

■ Tridiagonal Matrix

– The tridigonal matrix can be described as having a band width of 3.

– It can also be described as having a half-band width of 1, in reference to the number of nonzero diagonals on one side of the principal diagonal, that is, not including the diagonal where $i = j$ for a_{ij}.

Banded Matrices

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- Tridiagonal Matrix
  - The following relationship between the band width b_{w1} and half-band width b_{w2} can be obtained:

$$b_{w1`} = 2b_{w2} + 1$$

Banded Matrices

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- Tridiagonal Matrix
  - Banded matrices such as a tridigonal matrix have so many applications in engineering.
  - The can be stored more efficiently by storing the banded elements only, thereby reducing storage requirements for a solution.

Symmetric Matrices

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- Definition
  - A symmetric matrix is defined as a matrix where $a_{ij} = a_{ji}$.
  - Examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 3 & 0 \\ -1 & 4 & 5 & -2 \\ 3 & 5 & 7 & 4 \\ 0 & -2 & 4 & 15 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix}$$

Symmetric Matrices

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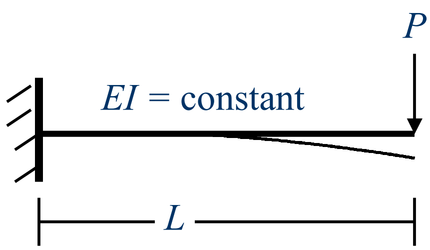
- Applications
  - In engineering, it is common to deal with symmetric matrices.
  - For example, the stiffness properties of a structural element can be described using symmetric *stiffness matrix*.
  - Also, correlations among the structural variables can be described using symmetric *correlation matrix*.

Symmetric Matrices

- Applications
  - Example: Stiffness Matrix for a Beam



$$S = \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Symmetric Matrices

- Properties
  - A symmetric matrix has the following properties:

$$A = A^T$$

  - The decomposition of ***A*** can be expressed as

$$A = LL^T$$

Symmetric Matrices

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■ Cholesky Decomposition Method

– This method uses a recurrence procedure to decompose a symmetric matrix into upper and lower triangular matrices.

– As a result, the ***LU*** decomposition can be computed more effectively.

Symmetric Matrices

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■ Cholesky Decomposition Method

$$l_{11} = \sqrt{a_{11}} \qquad \text{(1a)}$$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2} \qquad \text{for } i = 2,3,\cdots,n \qquad \text{(1b)}$$

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}} \qquad \text{for } j = 1,2,3,\cdots,i-1, \text{ and } j < i \qquad \text{(1c)}$$

Symmetric Matrices

■ Example:

Decompose the following matrix into its lower and upper triangular matrices using the Cholesky method:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{bmatrix}$$

Symmetric Matrices

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■ Example (cont'd)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{bmatrix}$$

$a_{11} = 1$
$a_{12} = 2 = a_{21}$
$a_{13} = 3 = a_{31}$
$a_{22} = 8$
$a_{23} = 10 = a_{32}$
$a_{33} = 22$

Using Eq. 1a:

$$l_{11} = \sqrt{a_{11}} = \sqrt{1} = 1$$

Using Eq. 1c:

$i = 2, j = 1$

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}} \qquad l_{21} = \frac{a_{21} - \sum_{k=1}^{1-1} l_{2k} l_{1k}}{l_{11}} = \frac{a_{21}}{l_{11}} = \frac{2}{1} = 2$$

Symmetric Matrices

■ Example (cont'd)

Using Eq. 1b:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{bmatrix}$$

$a_{11} = 1$
$a_{12} = 2 = a_{21}$
$a_{13} = 3 = a_{31}$
$a_{22} = 8$
$a_{23} = 10 = a_{32}$
$a_{33} = 22$
$l_{21} = 2$

$i = 2:$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$

$$l_{22} = \sqrt{a_{22} - \sum_{k=1}^{2-1} l_{2k}^2} = \sqrt{a_{22} - l_{21}^2} = \sqrt{8 - (2)^2} = 2$$

Symmetric Matrices

■ Example (cont'd)

Using Eq. 1c:

$i = 3, j = 1 \text{ and } 2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{bmatrix}$$

$a_{11} = 1$
$a_{12} = 2 = a_{21}$
$a_{13} = 3 = a_{31}$
$a_{22} = 8$
$a_{23} = 10 = a_{32}$
$a_{33} = 22$
$l_{21} = 2$
$l_{22} = 2$

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}}$$

$$l_{31} = \frac{a_{31} - \sum_{k=1}^{1-1} l_{3k} l_{1k}}{l_{11}} = \frac{a_{31}}{l_{11}} = \frac{3}{1} = 3$$

$$l_{32} = \frac{a_{32} - \sum_{k=1}^{2-1} l_{3k} l_{2k}}{l_{22}} = \frac{a_{32} - l_{31} l_{21}}{l_{22}} = \frac{10 - 3(2)}{2} = 2$$

Symmetric Matrices

- Example (cont'd)

Using Eq. 1b:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{bmatrix}$$

$a_{11} = 1$
$a_{12} = 2 = a_{21}$
$a_{13} = 3 = a_{31}$
$a_{22} = 8$
$a_{23} = 10 = a_{32}$
$a_{33} = 22$
$l_{21} = 2$
$l_{22} = 2$
$l_{31} = 3$
$l_{32} = 2$

$i = 3:$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$

$$l_{33} = \sqrt{a_{33} - \sum_{k=1}^{3-1} l_{3k}^2} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{22 - (3)^2 - (2)^2} = 3$$

Symmetric Matrices

- Example (cont'd)
  - Therefore

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 3 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$l_{11} = 1$
$l_{21} = 2$
$l_{22} = 2$
$l_{31} = 3$
$l_{32} = 2$
$l_{33} = 3$

Symmetric Matrices

- Example (cont'd)
  - Note that the validity of $\boldsymbol{LL^T} = \boldsymbol{A}$

$$LL^T = A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 10 \\ 3 & 10 & 22 \end{bmatrix}$$

Iterative Equation-Solving Methods

- Simultaneous equations can also be solved using *trial-and error* procedure.
- In this procedure, a solution can be assumed, that is, a set of estimates for the unknowns.
- Then, these estimates can be revised through some set of rules.

Iterative Equation-Solving Methods

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- This approach is the basis for iterative methods for solving simultaneous equations.
- Among these iterative methods, two procedures are considered:
  - Jacobi Iteration, and
  - Gauss-Seidel Iteration

Iterative Equation-Solving Methods

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- Jacobi Iteration
  - Consider the following general set of simultaneous equation:

$$\begin{array}{l} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n = C_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n = C_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \cdots + a_{3n}X_n = C_3 \\ \quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \cdots + a_{nn}X_n = C_n \end{array} \qquad (2)$$

Iterative Equation-Solving Methods

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■ Jacobi Iteration

– The first step in this method is rearrange each equation in Eq. 2 to produce an expression for a single unknown.

– To start the iterative calculations, an initial solution estimate for X_i unknowns is required.

Iterative Equation-Solving Methods

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■ Jacobi Iteration

$$
\begin{aligned}
X_1 &= \frac{C_1 - a_{12}X_2 - a_{13}X_3 - \cdots - a_{1n}X_n}{a_{11}} \\
X_2 &= \frac{C_2 - a_{21}X_1 - a_{23}X_3 - \cdots - a_{2n}X_n}{a_{22}} \\
&\vdots \\
X_n &= \frac{C_n - a_{n1}X_2 - a_{n2}X_3 - \cdots - a_{n-1,n}X_{n-1}}{a_{nn}}
\end{aligned}
\qquad (3)
$$

Iterative Equation-Solving Methods

- Jacobi Iteration
  - The initial estimates for all the X_i are substituted into the right sides of Eq. 3 to obtain a new set of calculated (left side) values for the X_i's.
  - These new values are substituted into the right side of Eq. 3 and a new set of values for the X_i's is obtained.
  - This iterative process continues until the calculated values for X_i's converge to an acceptable solution.

Iterative Equation-Solving Methods

- Example: Jacobi Iteration

  Solve the following set of equations using the Jacobi iterative method:

$$
\begin{aligned}
3X_1 + X_2 - 2X_3 &= 9 \\
-X + 4X_2 - 3X_3 &= -8 \\
X_1 - X_2 + 4X_3 &= 1
\end{aligned}
$$

Iterative Equation-Solving Methods

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■ Example (cont'd): Jacobi Iteration

The first step is to rearrange each equation as follows:

$$\begin{aligned} 3X_1 + X_2 - 2X_3 &= 9 \\ -X + 4X_2 - 3X_3 &= -8 \\ X_1 - X_2 + 4X_3 &= 1 \end{aligned} \Longrightarrow \begin{aligned} X_1 &= \frac{9 - X_2 + 2X_3}{3} \\ X_2 &= \frac{-8 + X_1 + 3X_3}{4} \\ X_3 &= \frac{1 - X_1 + X_2}{4} \end{aligned}$$

Iterative Equation-Solving Methods

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■ Example (cont'd): Jacobi Iteration

Let an estimate of the solution be $X_1 = X_2 = X_3 = 1$, therefore,

$$X_1 = \frac{9 - X_2 + 2X_3}{3} = \frac{9 - 1 + 2(1)}{3} = \frac{10}{3} = 3.333$$

$$X_2 = \frac{-8 + X_1 + 3X_3}{4} = \frac{-8 + 1 + 3(1)}{4} = -1$$

$$X_3 = \frac{1 - X_1 + X_2}{4} = \frac{1 - 1 + 1}{4} = \frac{1}{4} = 0.250$$

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Iterative Equation-Solving Methods

■ Example (cont'd): Jacobi Iteration

Revised values for X_i:
$X_1 = 3.333$
$X_2 = -1$
$X_3 = 0.25$

$$X_1 = \frac{9 - X_2 + 2X_3}{3} = \frac{9 - (-1) + 2(0.25)}{3} = 3.5$$

$$X_2 = \frac{-8 + X_1 + 3X_3}{4} = \frac{-8 + 3.3333 + 3(0.25)}{4} = -0.9792$$

$$X_3 = \frac{1 - X_1 + X_2}{4} = \frac{1 - (3.3333) + (-1)}{4} = -0.8333$$

Iterative Equation-Solving Methods

■ Example (cont'd): Jacobi Iteration

Revised values for X_i:
$X_1 = 3.5$
$X_2 = -0.9792$
$X_3 = -0.8333$

$$X_1 = \frac{9 - X_2 + 2X_3}{3} = \frac{9 - (-0.9792) + 2(-0.8333)}{3} = 2.7709$$

$$X_2 = \frac{-8 + X_1 + 3X_3}{4} = \frac{-8 + 3.5 + 3(-0.8333)}{4} = -1.7500$$

$$X_3 = \frac{1 - X_1 + X_2}{4} = \frac{1 - (3.5) + (-0.9792)}{4} = -0.8698$$

Iterative Equation-Solving Methods

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■ Example (cont'd): Jacobi Iteration

The solution converges to:
$X_1 = 3$
$X_2 = -2$
$X_3 = -1$

Iteration	X_1	$\|\Delta X_1\|$	X_2	$\|\Delta X_1\|$	X_3	$\|\Delta X_1\|$
0	1	-	1	-	1	-
1	3.3333	2.3333	-1.0000	2.0000	0.2500	0.7500
2	3.5000	0.1667	-0.9792	0.0208	-0.8333	1.0833
3	2.7708	0.7292	-1.7500	0.7708	-0.8698	0.0365
4	3.0035	0.2326	-1.9596	0.2096	-0.8802	0.0104
5	3.0664	0.0629	-1.9093	0.0503	-0.9908	0.1106
6	2.9759	0.0905	-1.9765	0.0672	-0.9939	0.0031
7	2.9962	0.0203	-2.0015	0.0250	-0.9881	0.0058
8	3.0084	0.0122	-1.9920	0.0094	-0.9994	0.0113
9	2.9977	0.0107	-1.9975	0.0054	-1.0001	0.0007
10	2.9991	0.0014	-2.0007	0.0032	-0.9988	0.0013
11	3.0010	0.0019	-1.9993	0.0013	-0.9999	0.0011
12	2.9998	0.0012	-1.9997	0.0004	-1.0001	0.0002
13	2.9998	0.0000	-2.0001	0.0004	-0.9999	0.0002
14	3.0001	0.0003	-1.9999	0.0002	-1.0000	0.0001
15	3.0000	0.0001	-2.0000	0.0000	-1.0000	0.0000

Iterative Equation-Solving Methods

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■ Gauss-Seidel Iteration

- Like in the Jacobi iteration, the first step in this method is rearrange each equation in Eq. 2 to produce an expression for a single unknown.
- To start the iterative calculations, an initial solution estimate for X_i unknowns is required.

Iterative Equation-Solving Methods

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■ Gauss-Seidel Iteration

$$X_1 = \frac{C_1 - a_{12}X_2 - a_{13}X_3 - \cdots - a_{1n}X_n}{a_{11}}$$

$$X_2 = \frac{C_2 - a_{21}X_1 - a_{23}X_3 - \cdots - a_{2n}X_n}{a_{22}} \qquad (4)$$

$$\vdots$$

$$X_n = \frac{C_n - a_{n1}X_2 - a_{n2}X_3 - \cdots - a_{n-1,n}X_{n-1}}{a_{nn}}$$

Iterative Equation-Solving Methods

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■ Gauss-Seidel Iteration

- In Jacobi iteration a full cycle is completed over all the equations before updating the solutions estimates.
- In Gauss-Seidel procedure, each unknown is updated as soon as a new estimate of that unknown is completed.
- The notion here is that the most recent estimate is the best estimate, and therefore, should be used as soon it is available.

Iterative Equation-Solving Methods

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■ Example: Gauss-Seidel Iteration

Solve the following system of equations using the Gauss-Seidel Iteration procedure with initial estimates of $X_1 = X_2 = X_3 = 1$:

$$
\begin{aligned}
4X - 2Y + 3Z &= 15.7 \\
-2X + 4Y - Z &= -14.1 \\
3X + Y - 3Z &= -4.2
\end{aligned}
$$

Iterative Equation-Solving Methods

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■ Example (cont'd): Gauss-Seidel Iteration

$$
\begin{aligned}
4X - 2Y + 3Z &= 15.7 \\
-2X + 4Y - Z &= -14.1 \\
3X + Y - 3Z &= -4.2
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
X &= \frac{15.7 + 2Y - 3Z}{4} \\
Y &= \frac{-14.1 + 2X + Z}{4} \\
Z &= \frac{-4.2 - 3X - Y}{-3}
\end{aligned}
$$

Iterative Equation-Solving Methods

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- Example (cont'd): Gauss-Seidel Iteration

    An estimate of the solution is $X_1 = X_2 = X_3 = 1$, therefore,

$$X = \frac{15.7 + 2Y - 3Z}{4} = \frac{15.7 + 2(1) - 3(1)}{4} = 3.6750$$

$$Y = \frac{-14.1 + 2X + Z}{4} = \frac{-14.1 + 2(3.6750) + (1)}{4} = -1.4375$$

$$Z = \frac{-4.2 - 3X - Y}{-3} = \frac{-4.2 - 3(3.6750) - (-1.4375)}{-3} = 4.5958$$

Iterative Equation-Solving Methods

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- Example (cont'd): Gauss-Seidel Iteration

Revised values for X_i:
$X = 3.6750$
$Y = -1.4375$
$Z = 4.5958$

$$X = \frac{15.7 + 2(-1.4375) - 3(4.5958)}{4} = -0.2406$$

$$Y = \frac{-14.1 + 2X + Z}{4} = \frac{-14.1 + 2(-0.2406) + 4.5958}{4} = -2.4964$$

$$Z = \frac{-4.2 - 3X - Y}{-3} = \frac{-4.2 - 3(-0.2406) - (-2.4964)}{-3} = 0.3273$$

Iterative Equation-Solving Methods

■ Example (cont'd): Gauss-Seidel Iteration

Revised values for X_i :
$X = -0.2406$
$Y = -2.4964$
$Z = 0.3273$

$$X = \frac{15.7 + 2Y - 3Z}{4} = \frac{15.7 + 2(-2.4964) - 3(0.3273)}{4} = 2.4313$$

$$Y = \frac{-14.1 + 2X + Z}{4} = \frac{-14.1 + 2(2.4313) + 0.3273}{4} = -2.2275$$

$$Z = \frac{-4.2 - 3X - Y}{-3} = \frac{-4.2 - 3(2.4313) - (-2.2275)}{-3} = 3.0888$$

Iterative Equation-Solving Methods

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■ Example (cont'd): Gauss-Seidel Iteration

The solution converges to:
X = 1.3
Y = -2.4
Z = 1.9

Iteration	X	$\lvert\Delta X\rvert$	Y	$\lvert\Delta Y\rvert$	Z	$\lvert\Delta Z\rvert$
0	1	-	1	-	1	-
1	3.6750	2.6750	-1.4375	2.4375	4.5958	3.5958
2	-0.2406	3.9156	-2.4964	1.0589	0.3273	4.2686
3	2.4314	2.6720	-2.2275	0.2689	3.0889	2.7616
4	0.4946	1.9368	-2.5055	0.2780	1.0594	2.0295
5	1.8777	1.3831	-2.3213	0.1842	2.5039	1.4445
6	0.8864	0.9913	-2.4558	0.1345	1.4678	1.0361
.	.	.	.	.	.	.
30	1.2999	0.0003	-2.4000	0.0000	1.8999	0.0003
31	1.3001	0.0002	-2.4000	0.0000	1.9001	0.0002
32	1.2999	0.0002	-2.4000	0.0000	1.8999	0.0002
33	1.3001	0.0001	-2.4000	0.0000	1.9001	0.0001
34	1.3000	0.0001	-2.4000	0.0000	1.9000	0.0001
35	1.3000	0.0001	-2.4000	0.0000	1.9000	0.0001
36	1.3000	0.0000	-2.4000	0.0000	1.9000	0.0000

Iterative Equation-Solving Methods

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- Convergence Consideration for the Iterative Methods
  - For these iterative techniques of Jacobi and Gauss-Seidel to work, certain additional conditions must be considered:
    1. The set of equations must possess a strong diagonal.
    2. A sufficient condition for a solution to be found is that the absolute value of the diagonal coefficient in any equation must greater than the sum of the absolute values of all other coefficients appearing in that equation.

Iterative Equation-Solving Methods

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- Convergence Consideration for the Iterative Methods
  - Before solving a set of equations using the iterative methods, do the following:
    - Rearrange the set of $n \times n$ equations so that the diagonal coefficient is the largest in any equation.

Iterative Equation-Solving Methods

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■ Example: Convergence

Check the following set of equations for convergence. If they do not meet the condition for convergence, try to rearrange the equation so that they meet the requirement.

$$\begin{aligned} X_1 + 4X_2 - 2X_3 &= 3 \\ 5X_1 - 2X_2 + X_3 &= 4 \\ X_1 + 2X_2 + 4X_3 &= 17 \end{aligned}$$

Iterative Equation-Solving Methods

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■ Example (cont'd): Convergence

$$\begin{aligned} X_1 + 4X_2 - 2X_3 &= 3 \\ 5X_1 - 2X_2 + X_3 &= 4 \\ X_1 + 2X_2 + 4X_3 &= 17 \end{aligned} \longrightarrow \begin{aligned} 5X_1 - 2X_2 + X_3 &= 4 \\ X_1 + 4X_2 - 2X_3 &= 3 \\ X_1 + 2X_2 + 4X_3 &= 17 \end{aligned}$$

$|1| < |4| + |-2| \Rightarrow 1 < 6$ N.G

Try rearrange the equations.

$|5| > |-2| + |1| \Rightarrow 5 > 3$ O.K

$|4| > |1| + |-2| \Rightarrow 4 > 3$ O.K.

$|4| > |1| + |2| \Rightarrow 4 > 3$ O.K.

Therefore, use the above set in iterative methods.

Use of Determinants

- As was noted earlier, a determinant is a unique number that can be used to represent a square matrix.
- For the matrix ***A***, the determinant is denoted as |***A***|.
- Recall that the system of equations is given by [***A***]{***X***} = {***C***}

Use of Determinants

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- Or in equivalent general matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} \qquad (5)$$

Use of Determinants

- In which, the coefficient matrix ***A*** is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n} \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix}, \text{and } \; C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}$$

X = vector of unknowns, and ***C*** = vector of constants

Use of Determinants

- Cramer's Rule

  The value of X_i is obtained using Cramer's rule as

$$X_i = \frac{|A_i|}{|A|}$$

Where |***A***| is the determinant of ***A*** and |$\boldsymbol{A_i}$| is the determinant of a matrix formed by replacing column *I* of ***A*** with the column vector of constant of Eq.5.

Use of Determinants

- Cramer's Rule
  - For example, $|\boldsymbol{A}_1|$ and $|\boldsymbol{A}_2|$ would be given by

$$|A_1| = \begin{vmatrix} C_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ C_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \text{ and } |A_2| = \begin{vmatrix} a_{11} & C_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & C_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & C_3 & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

Use of Determinants

- Example: Use of Determinant

  Solve the following set of simultaneous linear equations using the method of the determinants:

$$4X - 2Y + 3Z = 15.7$$
$$-2X + 4Y - Z = -14.1$$
$$3X + Y - 3Z = -4.2$$

Use of Determinants

■ Example (cont'd): Use of Determinant

$$[A]\{X\}=\{C\}$$

$$\begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & 1 & -3 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}=\begin{bmatrix} 15.7 \\ -14.1 \\ -4.2 \end{bmatrix}$$

$$A=\begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & 1 & -3 \end{bmatrix} \quad \text{and} \quad C=\begin{bmatrix} 15.7 \\ -14.1 \\ -4.2 \end{bmatrix}$$

Use of Determinants

■ Example (cont'd): Use of Determinant

$$|A|=\begin{vmatrix} 4 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & 1 & -3 \end{vmatrix}=4[4(-3)-1(-1)]+2[-2(-3)-3(-1)]+3[-2(1)-3(4)]=-68$$

$$\boxed{A=\begin{bmatrix} \cancel{4} & -2 & 3 \\ \cancel{-2} & 4 & -1 \\ \cancel{3} & 1 & -3 \end{bmatrix} \quad \text{and} \quad C=\begin{bmatrix} 15.7 \\ -14.1 \\ -4.2 \end{bmatrix}}$$

$$X=\frac{|A_1|}{|A|}=\frac{\begin{vmatrix} 15.7 & -2 & 3 \\ -14.1 & 4 & -1 \\ -4.2 & 1 & -3 \end{vmatrix}}{-68}=\frac{-88.4}{-68}=1.3$$

Use of Determinants

■ Example (cont'd): Use of Determinant

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & 1 & -3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 15.7 \\ -14.1 \\ -4.2 \end{bmatrix}$$

$$Y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 4 & 15.7 & 3 \\ -2 & -14.1 & -1 \\ 3 & -4.2 & -3 \end{vmatrix}}{-68} = \frac{163.2}{-68} = -2.4$$

Use of Determinants

■ Example (cont'd): Use of Determinant

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & 1 & -3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 15.7 \\ -14.1 \\ -4.2 \end{bmatrix}$$

$$Z = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 4 & -2 & 15.7 \\ -2 & 4 & -14.1 \\ 3 & 1 & -4.2 \end{vmatrix}}{-68} = \frac{-129.2}{-68} = 1.9$$

Use of Determinants

- Example (cont'd): Use of Determinant

Therefore, the solution for set of equation

$$\begin{aligned} 4X - 2Y + 3Z &= 15.7 \\ -2X + 4Y - Z &= -14.1 \\ 3X + Y - 3Z &= -4.2 \end{aligned}$$

is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1.3 \\ -2.4 \\ 1.9 \end{bmatrix}$$