CHAPTER 5a.
SIMULTANEOUS LINEAR EQUATIONS

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Introduction

- Systems of simultaneous equations can be found in many engineering applications and problems.
- Systems that consist of small number of equations can be solved analytically using standard methods from algebra.
- Systems of large number of equations require the use of numerical methods and computers.
Introduction

Simultaneous Linear Equations and Engineering

- The system of simultaneous equations is probably one of the most important topics in modern engineering computations.
- This is not an exaggeration if one considers that recent technological advances were made possible by the ability of solving larger and larger systems of equations.

<table>
<thead>
<tr>
<th>Small Systems</th>
<th>Larger System</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 + 3X_2 + 2X_3 = 15 )</td>
<td>( X_1 + 2X_2 - 4X_3 + \cdots + 10X_{20} = 20.1 )</td>
</tr>
<tr>
<td>( 2X_1 + 4X_2 + 3X_3 = 22 )</td>
<td>( -2X_1 + X_2 - 3X_3 + \cdots + 2X_{20} = 2 )</td>
</tr>
<tr>
<td>( 3X_1 + 4X_2 + 7X_3 = 39 )</td>
<td>( 2X_1 - 7X_2 - 4X_3 + \cdots - 8X_{20} = -6.5 )</td>
</tr>
<tr>
<td>( 3X - 0.1Y - 0.2Z = 7.85 )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( 0.1X + 7Y - 0.3Z = -19.2 )</td>
<td>( 8X_1 - 2X_2 + 8X_3 + \cdots + 3X_{20} = -11 )</td>
</tr>
<tr>
<td>( 0.3X - 0.2Y + 10Z = 71.7 )</td>
<td></td>
</tr>
</tbody>
</table>
Introduction

Simultaneous Linear Equations and Engineering

- Technological advances in engineering that involve large number of simultaneous equations include:
  - The finite element method
  - The finite difference method
  - The analysis of structural, mechanical, and electrical systems.

In the past, the numerical methods (such as the finite element and finite difference methods) that were used to solve systems of simultaneous equations were not attractive owing to the tremendous amount of calculations involved.

However, computers have changed that and altered our approach to engineering problem solving.
Introduction

■ Engineering Examples

– Electrical Circuit

$I = \text{current} \quad V = \text{voltage} \quad R = \text{resistance}$

 Kirchhoff’s first law states that the algebraic sum of current flowing into a junction of a circuit must equal zero

 Kirchhoff’s second law states that the algebraic sum of the electromotive forces around a closed circuit must equal the sum of voltage drops around the circuit.
Introduction

Engineering Examples

- Electrical Circuit

Applying Kirchhoff’s first law at junction c, yields the following linear equation:

\[ I_1 + I_2 - I_3 = 0 \]

Applying Kirchhoff’s second law to loop \( acdb \) yields the following linear equation:

\[ V_1 = R_1 I_1 + R_3 I_3 \]
Introduction

■ Engineering Examples
  – Electrical Circuit
    • Applying Kirchhoff’s second law to loop aefb yields the following linear equation:

\[ V_1 - V_2 = R_1 I_1 - R_2 I_2 \]

If \( R_1 = 2, R_2 = 4, R_3 = 5, V_1 = 6, \) and \( V_2 = 2, \) then

\[
\begin{align*}
I_1 + I_2 - I_3 &= 0 \\
2I_1 + 5I_3 &= 6 \\
2I_1 - 4I_2 &= 4
\end{align*}
\]

The solution to these three equations produces the current flows in the network.
Introduction

Engineering Examples
– Analysis of Statically Determinant Truss

\[ \sum F_H = H_2 = 0 \]
\[ \sum F_V = V_1 + V_2 - 1000 = 0 \]
**Introduction**

**Engineering Examples**

- **Analysis of Statically Determinant Truss**

\[ \sum F_H = -F_1 \cos 30 + F_3 \cos 60 = 0 \]
\[ \sum F_V = -1000 - F_1 \sin 30 - F_3 \sin 60 = 0 \]  

(1)

\[ \sum F_H = H_2 + F_1 \cos 30 + F_2 = 0 \]
\[ \sum F_V = V_2 + F_1 \sin 30 = 0 \]  

(2)
Introduction

Engineering Examples

– Analysis of Statically Determinant Truss

\[ \sum F_H = -F_2 - F_3 \cos 60 = 0 \]
\[ \sum F_V = V_3 + F_3 \sin 60 = 0 \]

\[ \sum F_H = -F_2 - F_3 \cos 60 = 0 \]
\[ \sum F_V = V_3 + F_3 \sin 60 = 0 \]

\[ \sum F_H = H_2 + F_1 \cos 30 + F_2 = 0 \]
\[ \sum F_V = V_2 + F_1 \sin 30 = 0 \]
## Engineering Examples

### Analysis of Statically Determinant Truss

- The solution to the previous system of equations provides the following force values for the truss:

<table>
<thead>
<tr>
<th>Force</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>$-500$ lb</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$433$ lb</td>
</tr>
<tr>
<td>$F_3$</td>
<td>$-866$ lb</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$0$ (as expected)</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$250$ lb</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$750$ lb</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
-0.866F_1 + 0.5F_3 &= 0 \\
-0.5F_1 - 0.866F_3 &= 1000 \\
0.866F_1 + F_2 + H_2 &= 0 \\
0.5F_1 + V_2 &= 0 \\
-F_2 - 0.5F_3 &= 0 \\
0.866F_3 + V_3 &= 0
\end{align*}
\]
Introduction

- Engineering Examples
  - Engineering Dynamics Example:
    • Suppose that a team of three parachutists is connected by a weightless cord while free-falling at a velocity of 5 m/s as shown in the figure. Compute the tension in each section of the cord and the acceleration of the team, given the masses of each parachutist and the drag coefficients as provided in the table.

<table>
<thead>
<tr>
<th>Parachutist</th>
<th>Mass (kg)</th>
<th>Drag Coefficient (kg/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>70</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>18</td>
</tr>
</tbody>
</table>
Introduction

Engineering Examples

– Engineering Dynamics Example

Free-body diagrams are needed for each of the parachutists as shown in the next viewgraph.

Summing forces in the vertical direction and using Newton’s second law of motion, gives:

\[
\begin{align*}
    m_1g - T - c_1v &= m_1a \\
    m_2g + T - c_2v - R &= m_2a \\
    m_3g - c_3v + R &= m_3a
\end{align*}
\]
Introduction

Engineering Examples

- **Engineering Dynamics Example**

Substituting the values for parachutists' masses and drag coefficients, the system of equations provided by Eq. 4, gives

\[
\begin{align*}
80(9.8) - T - 11(5) &= 80a \\
70(9.8) + T - 15(5) - R &= 70a \\
50(9.8) - 18(5) + R &= 50a
\end{align*}
\]
Introduction

Engineering Examples

- Engineering Dynamics Example

After rearranging and simplifying,

\[ 80a + T = 729 \]
\[ 70a - T + R = 611 \]
\[ 50a - R = 400 \]

The solution to the above system of equations gives the following values:

\[ a = 8.7 \text{ m/s}^2, \quad T = 33 \text{ N}, \quad \text{and } R = 35 \text{ N} \]

General Form for a System of Equations

- Definition

A linear equation is one in which a variable only appears to the first power in every term of a given equation.

Thus, a system of \( m \) linear equations in \( n \) unknowns \( X_j, j = 1, 2, \ldots, n \), can be represented as

\[ \sum_{j=1}^{n} a_j X_j = C_i, \quad i = 1, 2, \ldots, m \]
General Form for a System of Equations

**Expanded Form**

\[
\begin{align*}
 a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= C_1 \\
 a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= C_2 \\
 \quad & \vdots \\
 a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= C_m
\end{align*}
\]  

\[a_{ij} = \text{known coefficients of the equations} \]  
\[X_j = \text{unknown variables} \]  
\[C_i = \text{known constants} \]

**Expanded Form (m = n)**

\[
\begin{align*}
 a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= C_1 \\
 a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= C_2 \\
 \quad & \vdots \\
 a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n &= C_n
\end{align*}
\]  

\[a_{ij} = \text{known coefficients of the equations} \]  
\[X_j = \text{unknown variables} \]  
\[C_i = \text{known constants} \]
General Form for a System of Equations

Classification of Systems of Equations

1. A set of equations in which the number of unknowns is equal to the number of equations \((n = m)\)
2. A set of equations in which the number of unknowns is less than the number of equations \((n < m)\)
3. A set of equations in which the number of unknowns is greater than the number of equations \((n > m)\)

Solution of a System of Two Equations

The solution of two equations gives insight to understand the classification of systems of equations based on graphical interpretation in two-dimensional space.

- If \(n\) equals 2, then Eq. 5 reduces to

\[
\begin{align*}
    a_{11}X_1 + a_{12}X_2 &= C_1 \quad (6a) \\
    a_{21}X_1 + a_{22}X_2 &= C_2 \quad (6b)
\end{align*}
\]
Solution of a System of Two Equations

– A solution for simple system can be obtained by substitution.
– Solving Eq. 6a for $X_1$ gives

$$X_1 = \frac{C_1 - a_{12}X_2}{a_{11}} \quad (7a)$$

– The expression for $X_1$ in the above equation can be substituted into Eq. 6b:

$$a_{21} \left( \frac{C_1 - a_{12}X_2}{a_{11}} \right) + a_{22}X_2 = C_2 \quad (7b)$$

Solution of a System of Two Equations

– Equation 7b is a single equation with one unknown, $X_2$.
– This equation can be solved for $X_2$ to give

$$X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} \quad (8a)$$

– Eq. 8a can be substituted back into Eq. 7a to give

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} \quad (8b)$$
Solution of a System of Two Equations

- It seems that the solution procedure for this set of equations is simple to apply.
- One should imagine the effort that would be required to solve 15 or 20 simultaneous equations using the substitution procedure.
- Many complex engineering problems involve hundreds or even thousands of simultaneous equations.
- Hence, the need for alternative solution procedures is justified.

Types of Numerical Procedures

- Because of the wide-spread use of computers nowadays, numerical solution methods are widely used.
  There are three general types:
  1. Elimination methods,
  2. Iteration methods, and
Classification of Systems of Equations
Based on Graphical Interpretation

Systems of equations can be classified based on their solutions to the following types:

1. Systems that have solutions,
2. Systems without solution, and
3. Systems with an infinite number of solutions.