CHAPTER 4d.
ROOTS OF EQUATIONS

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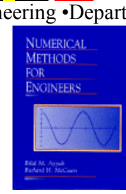

by

Dr. Ibrahim A. Assakkaf

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering
University of Maryland, College Park

Newton-Raphson Method

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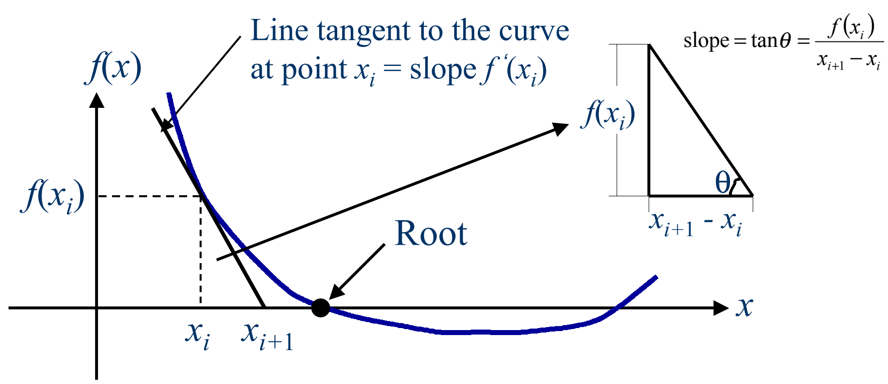
- Although the bisection method will always converge on the root, the rate of convergence is very slow.
- A faster method for converging on a single root of a function is the Newton-Raphson method.
- Perhaps it is the most widely used method of all locating formulas.

Newton-Raphson Method

■ Derivation of Newton-Raphson Method



Newton-Raphson Method

■ Derivation of Newton-Raphson Method

– <u>Graphical Derivation</u>

From the previous figure,

$$\text{Slope} = -f'(x_i) = \left.\frac{df(x)}{dx}\right|_{x=x_i} = \frac{f(x_i) - 0}{x_{i+1} - x_i}$$

or

$$x_{i+1} - x_i = \frac{f(x_i)}{-f(x_i)}$$

or

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Newton-Raphson Method

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■ Derivation of Newton-Raphson Method

– Derivation using Taylor Series

Recall Taylor series expansion,

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2!}f^{(2)}(x_0)+\frac{h^3}{3!}f^{(3)}(x_0)+\ldots+\frac{h^n}{n!}f^{(n)}(x_0)+R_{n+1}$$

If we let $x_0 + h = x_i + h = x_{i+1}$ and terminate the series at its linear term, then

$$f(x_i+h)=f(x_i)+(x_{i+1}-x_i)f^{(1)}(x_i)$$

or

$$f(x_{i+1})=f(x_i)+(x_{i+1}-x_i)f'(x_i)$$

Newton-Raphson Method

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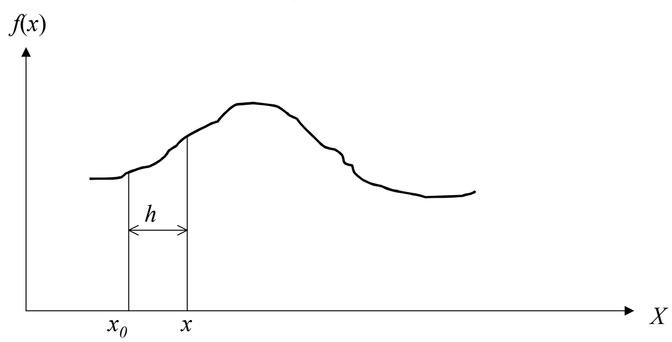
■ Derivation of Newton-Raphson Method

– Derivation using Taylor Series



Newton-Raphson Method

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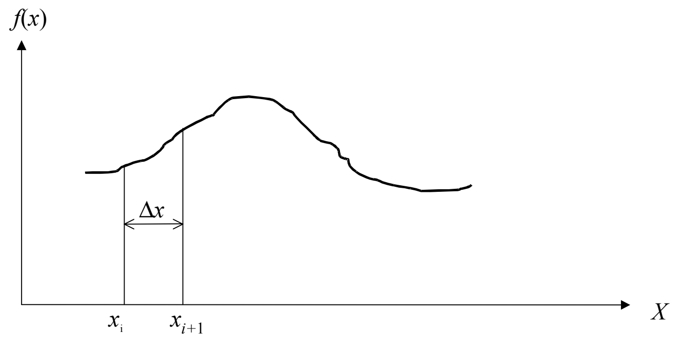
■ Derivation of Newton-Raphson Method

– Derivation using Taylor Series



Newton-Raphson Method

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■ Derivation of Newton-Raphson Method

Note that since the root of the function relating $f(x)$ and x is the value of x when $f(x_{i+1}) = 0$ at the intersection, hence,

$$f(x_{i+1}) = 0 = f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

or

$$(x_{i+1} - x_i) f'(x_i) = -f(x_i)$$

or

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Newton-Raphson Method

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■ Newton-Raphson Iteration

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

where

x_i = value of the root at iteration i

x_{i+1} = a revised value of the root at iteration i +1

$f(x_i)$ = value of the function at iteration i

$f(x_i)$ = derivative of $f(x)$ evaluated at iteration i

Newton-Raphson Method

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■ Example 1

Use the Newton-Raphson iteration method to estimate the root of the following function employing an initial guess of x_0 =0:

$$f(x) = e^{-x} - x$$

Let's find the derivative of the function first,

$$f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$$

Newton-Raphson Method

■ Example 1 (cont'd)

The initial guess is $x_0 = 0$, hence,

$i = 0$:

$f(0) = e^{-(0)} - 0 = 1$

$f'(0) = -e^{-(0)} - 1 = -1-1=-2$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{-2} = 0.5$$

$$f(x) = e^{-x} - x$$

$$f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$$

Newton-Raphson Method

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■ Example 1 (cont'd)

Now $x_1 = 0.5$, hence,

$i = 1$

$f(0) = e^{-(0.5)} – (0.5) = 0.1065$

$f'(0) = -e^{-(0.5)} - 1 = -1.6065$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{0.1065}{-1.6065} = 0.5663$$

$$f(x) = e^{-x} - x$$

$$f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$$

Newton-Raphson Method

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- Example 1 (cont'd) $f(x) = e^{-x} - x$

  Now $x_2 = 0.5663$, hence, $f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$

  $\underline{i = 2}$

  $f(0) = e^{-(0.5663)} - (0.5663) = 0.001322$

  $f'(0) = -e^{-(0.5663)} - 1 = -1.567622$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5663 - \frac{0.001322}{-1.567622} = 0.5671$$

Newton-Raphson Method

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- Example 1 (cont'd) $f(x) = e^{-x} - x$

  Now $x_3 = 0.5671$, hence, $f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$

  $\underline{i = 3}$

  $f(0) = e^{-(0.5671)} - (0.5671) = 0.00006784$

  $f'(0) = -e^{-(0.5671)} - 1 = -1.56716784$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.5671 - \frac{0.00006784}{-1.56716784} = 0.5671$$

Newton-Raphson Method

■ Example 1 (cont'd)

Thus, the approach rapidly converges on the true root of 0.5671 to four significant digits.

i	x_i	$f(x_i)$	$f'(x_i)$	Percent $\lvert\varepsilon_r\rvert$
0	0	1	-2	---
1	0.5	0.106531	-1.6065307	100
2	0.566311003	0.001305	-1.5676155	11.709291
3	0.567143165	1.96E-07	-1.5671434	0.14672871
4	0.56714329	4.44E-15	-1.5671433	2.2106E-05
5	0.56714329	0	-1.5671433	5.0897E-13

Hence, the root is 0.5671.

Newton-Raphson Method

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■ Example 2

The following polynomial has a root within the interval $3.75 \le x \le 5.00$:

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

If a tolerance of 0.001 (0.1%) is required, find this root using both the bisection and Newton-Raphson methods. Compare the rate of convergence on the root between the two methods.

Newton-Raphson Method

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■ Example 2 (cont'd)

Bisection Method:

$$\boxed{f(x) = x^3 - x^2 - 10x - 8 = 0}$$

$x_s = 3.75, \qquad x_e = 5.00$

$i = 1$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 5.00}{2} = 4.375$$

$$f(x_s) = f(3.75) = (3.75)^3 - (3.75)^2 - 10(3.75) - 8 = -6.828$$

$$f(x_m) = f(4.375) = (4.375)^3 - (4.375)^2 - 10(4.375) - 8 = 12.850$$

$$f(x_e) = f(5) = (5)^3 - (5)^2 - 10(5) - 8 = 42.000$$

$f(x_s)f(x_m) < 0$ (negative)

$f(x_m)f(x_e) > 0$ (positive)

Newton-Raphson Method

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■ Example 2 (cont'd)

Bisection Method:

$$\boxed{f(x) = x^3 - x^2 - 10x - 8 = 0}$$

$x_s = 3.75 \qquad x_e = 4.375$

$i = 2$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 4.375}{2} = 4.063$$

$f(x_s) = f(3.75) = -6.828$

$f(x_m) = f(4.063) = 1.918$

$f(x_e) = f(4.375) = 12.850$

$f(x_s)f(x_m) < 0$ (negative)

Newton-Raphson Method

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■ Example 2 (cont'd)

Bisection Method:

Iteration i	x_s	x_m	x_e	$f(x_s)$	$f(x_m)$	$f(x_e)$	$f(x_s)f(x_m)$	$f(x_m)f(x_e)$	error ε_d	error ε_d
1	3.7500	4.3750	5.0000	-6.8281	12.8496	42.0000	-	+	---	---
2	3.7500	4.0625	4.3750	-6.8281	1.9182	12.8496	-	+	0.31250	7.69
3	3.7500	3.9063	4.0625	-6.8281	-2.7166	1.9182	+	-	0.15625	4.00
4	3.9063	3.9844	4.0625	-2.7166	-0.4661	1.9182	+	-	0.07813	1.96
5	3.9844	4.0234	4.0625	-0.4661	0.7092	1.9182	-	+	0.03906	0.97
6	3.9844	4.0039	4.0234	-0.4661	0.1174	0.7092	-	+	0.01953	0.49
7	3.9844	3.9941	4.0039	-0.4661	-0.1754	0.1174	+	-	0.00977	0.24
8	3.9941	3.9990	4.0039	-0.1754	-0.0293	0.1174	+	-	0.00488	0.12
9	3.9990	4.0015	4.0039	-0.0293	0.0440	0.1174	-	+	0.00244	0.06
10	3.9990	4.0002	4.0015	-0.0293	0.0073	0.0440	-	+	0.00122	0.03
11	3.9990	3.9996	4.0002	-0.0293	-0.0110	0.0073	+	-	0.00061	0.02
12	3.9996	3.9999	4.0002	-0.0110	-0.0018	0.0073	+	-	0.00031	0.01
13	3.9999	4.0001	4.0002	-0.0018	0.0027	0.0073	-	+	0.00015	0.00
14	3.9999	4.0000	4.0001	-0.0018	0.0005	0.0027	-	+	0.00008	0.00
15	3.9999	4.0000	4.0000	-0.0018	-0.0007	0.0005	+	-	0.00004	0.00

Newton-Raphson Method

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■ Example 2 (cont'd)

$$f(x) = x^3 - x^2 - 10x - 8$$

Newton-Raphson Iteration:

$$f'(x) = 3x^2 - 2x - 10$$

The initial guess is x_0 =3.75 , hence,

$i = 0$:

$f(3.75) = (3.75)^3 - (3.75)^2 - 10(3.75) - 8 = -6.8281$

$f'(3.75) = 3(3.75)^2 - 2(3.75) - 10 = 24.6875$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.75 - \frac{-6.8281}{24.6875} = 4.0266$$

Newton-Raphson Method

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■ Example 2 (cont'd)

$f(x) = x^3 - x^2 - 10x - 8$

Newton-Raphson Iteration:

$f'(x) = 3x^2 - 2x - 10$

Now we have x_1 =4.0266 , hence,

$i = 1$:

$f(4.0266) = 0.8052$

$f'(4.0266) = 30.5869$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.0266 - \frac{0.8052}{30.5869} = 4.0003$$

Newton-Raphson Method

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■ Example 2 (cont'd)

i	x_i	$f(x_i)$	$f'(x_i)$	$\lvert\varepsilon_d\rvert$	Percent $\lvert\varepsilon_r\rvert$
0	3.75	-6.8281	24.688	---	---
1	4.0266	0.8053	30.587	0.2766	6.87
2	4.0003	0.0077	30.006	0.0263	0.66
3	4	7E-07	30	0.0003	0.01
4	4	3E-14	30	0.0000	0.00
5	4	0	30	0.0000	0.00

The rate of convergence with Newton-Raphson iteration is much faster than the bisection method.
N-R method converges to the exact root in 3 iterations.

Newton-Raphson Method

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■ Pitfalls of the Newton-Raphson Method

– Nonconvergence

- Nonconvergence can occur if the initial estimate is selected such that the derivative of the function equals zero.
- In such case, $f'(x_i)$ would be zero and $f(x_i) / f'(x_i)$ would go to infinity.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{f(x_i)}{0} \Rightarrow \infty$$

Newton-Raphson Method

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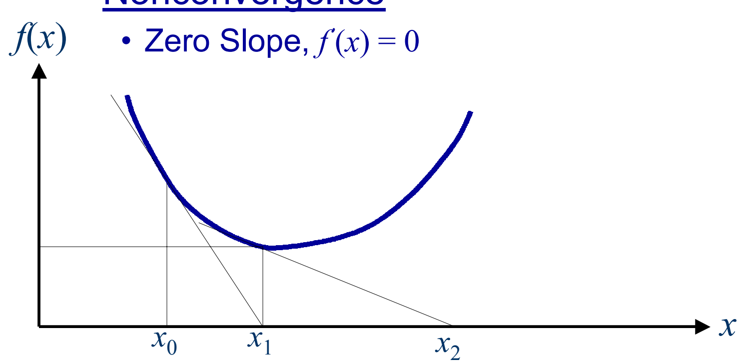
■ Pitfalls of the Newton-Raphson Method

– Nonconvergence

- Zero Slope, $f'(x) = 0$

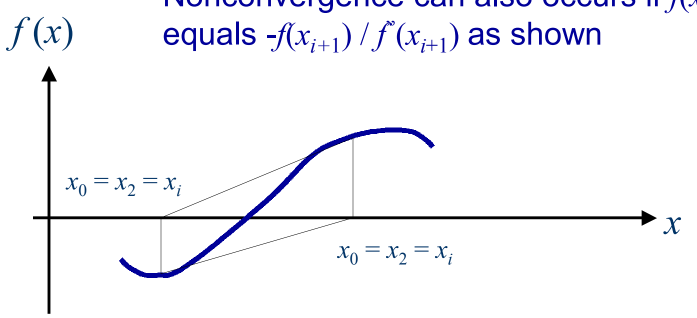


Newton-Raphson Method

■ Pitfalls of the Newton-Raphson Method

– Nonconvergence

• Nonconvergence can also occurs if $f(x_i) / f'(x_i)$ equals $-f(x_{i+1}) / f'(x_{i+1})$ as shown



Newton-Raphson Method

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■ Pitfalls of the Newton-Raphson Method

– Excessive Iteration

• A large number of iterations will be required if the value of $f'(x_i)$ is much larger than $f(x_i)$.

• In this case, $f(x_i) / f'(x_i)$ is small, which leads to a smaller adjustment at each iteration.

• This situation can occur, for example, when the root of a polynomial is near zero.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \text{small number}$$

Secant Method

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- A potential problem in utilizing Newton-Raphson method is the evaluation of the derivative.
- Although this is not true for polynomials and many other functions, there are certain functions whose derivatives may be extremely difficult or inconvenient to evaluate.

Secant Method

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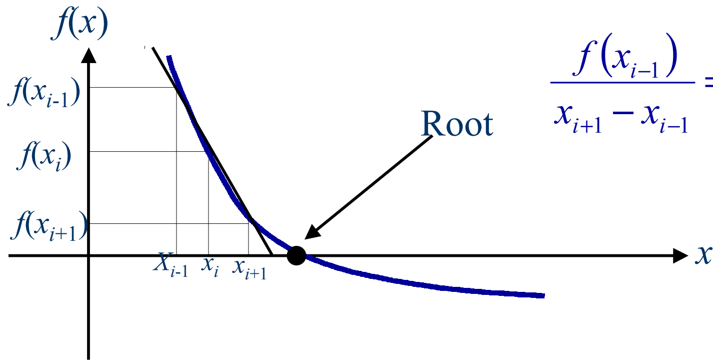
- The secant method is similar to the Newton-Raphson method with the difference that the derivative $f'(x)$ is numerically evaluated, rather computed analytically.

Secant Method

■ Development of the Secant Method



$$\frac{f(x_{i-1})}{x_{i+1}-x_{i-1}} = \frac{f(x_i)}{x_{i+1}-x_i}$$

Secant Method

■ Development of the Secant Method

– Using the geometric similarities of two triangles of the previous figure, Hence

$$\frac{f(x_{i-1})}{x_{i+1}-x_{i-1}} = \frac{f(x_i)}{x_{i+1}-x_i}$$

or

$$x_{i+1} = x_i - \frac{f(x_i)[x_{i-1}-x_i]}{f(x_{i-1})-f(x_i)}$$

Secant Method

■ The Secant Method

A new estimate of the root can be obtained using values of the function $f(x_i)$ and $f(x_{i-1})$ at two other estimates x_i and x_{i-1} of the root, and applying the following iterative procedure:

$$x_{i+1} = x_i - \frac{f(x_i)[x_{i-1} - x_i]}{f(x_{i-1}) - f(x_i)}$$

Secant Method

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■ Example 1

Use the secant method to estimate the root of the following function:

$$f(x) = e^{-x} - x$$

Start with initial estimates of $x_{i-1} = 0$ and $x_i = 1$.

Secant Method

■ Example 1 (cont'd)

First iteration, *i* = 1:

$$x_0 = 0 \Rightarrow f(0) = e^{-(0)} - (0) = 1$$
$$x_1 = 1 \Rightarrow f(1) = e^{-(1)} - 1 = -0.63212$$
$$x_2 = x_1 - \frac{f(x_1)[x_0 - x_1]}{f(x_0) - f(x_1)} = 1 - \frac{-0.63212[0-1]}{1-(-0.63212)} = 0.61270$$

Secant Method

■ Example 1 (cont'd)

Second iteration, *i* = 2:

$$x_1 = 1, \Rightarrow f(x_1) = -0.63212$$
$$x_2 = 0.61270, \Rightarrow f(0.61270) = -0.07081$$
$$x_3 = x_2 - \frac{f(x_2)[x_1 - x_2]}{f(x_1) - f(x_2)} = 0.61270 - \frac{-0.07081[1-0.61270]}{-0.63212-(-0.07081)} = 0.56384$$

Secant Method

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- Example 1 (cont'd)

  Third iteration, i = 3:

$$x_2 = 0.61270, \Rightarrow f(x_1) = -0.07081$$

$$x_3 = 0.56384, \Rightarrow f(0.56384) = 0.00518$$

$$x_4 = x_3 - \frac{f(x_3)[x_2 - x_3]}{f(x_2) - f(x_3)} = 0.56384 - \frac{0.00518[0.61270 - 0.56384]}{-0.07081 - 0.00518} = 0.56717$$

$$f(0.56717) = -0.00004$$

Hence, the root is 0.56717 to 4 significant digits.

Polynomial Reduction

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- After one root of a polynomial has been found, the process can be repeated using a new estimate.
- However, if proper consideration is not given to the selection of the new initial estimate of the second root, then application of some method might result in the same root being found.

Polynomial Reduction

■ Definition

Polynomial reduction states that if the polynomial $f(x)$ equals zero and root x_1 is the root of $f(x)$, then there is a reduced polynomial $f^*(x)$ such that $(x - x_1)\, f^*(x) = 0$, where

$$f^*(x) = \frac{f(x)}{x - x_1}$$

If $f(x)$ is a polynomial of order n, the reduced polynomial is of order $n - 1$.

Polynomial Reduction

■ Example

Using Newton-Raphson iteration, a root of $x_1 = 4$ was found for the following polynomial: $x^3 - x^2 - 10x - 8$. Reduce this polynomial.

$$
\begin{array}{r|l}
 & x^2 + 3x^2 + 2 \\
x - 4 & x^3 - x^2 - 10x - 8 \\
 & \underline{x^3 - 4x^2} \\
 & \quad 3x^2 - 10x \\
 & \quad \underline{3x^2 - 12x} \\
 & \qquad 2x - 8 \\
 & \qquad \underline{2x - 8} \\
 & \qquad\quad 0 = \text{error}
\end{array}
$$

Polynomial Reduction

- Example
  - The reduced polynomial $x^2 + 3x^2 + 2$ can be used to find additional roots for the original polynomial $x^3 - x^2 - 10x - 8$.
  - Any other method then can be used to find a root of the reduced polynomial, and the polynomial can be reduced again using polynomial reduction until all of the roots are found.