CHAPTER 4c.
ROOTS OF EQUATIONS

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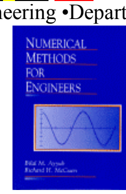

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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Bisection Method

- The bisection method or <u>*interval-halving*</u> is an extension of the direct-search method.
- It is used in cases where it is known that only one root occurs within a given interval of x.
- For the same level of precision, this method requires fewer calculations than the direct search method.

Bisection Method

- Underlying Concept
  - The bisection (interval-halving) method is one of the simplest technique for determining roots of a function.
  - The basis for this method can be easily illustrated by considering the following function:

$$y = f(x)$$

Bisection Method

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- Underlying Concept
  - One objective is to find an x value for which y is zero.
  - Using this method, the function can be evaluated at two x values, say x_1 and x_2 such that

$$f(x_1)f(x_2) < 0$$

Bisection Method

■ Underlying Concept

– The implication is that one of the values is negative and the other is positive.

– Also, the function must be continuous for

$$x_1 \le x \le x_2$$

– These conditions can be easily satisfied by sketching the function as shown in the following figure:

Bisection Method

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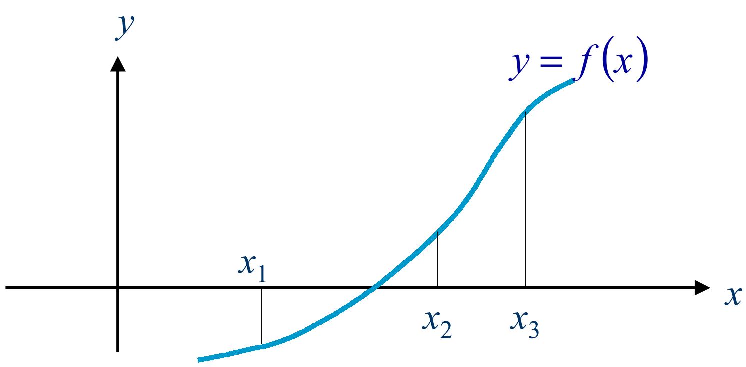
■ Underlying Concept



Bisection Method

- Underlying Concept
  - Looking at the figure, it is clear that the function is negative at x_1 and positive at x_2, and is continuous for $x_1 \leq x \leq x_2$.
  - Therefore, the root must between x_1 and x_2 and a new approximation to the root can be calculated as

$$x_3 = \frac{x_1 + x_2}{2}$$

Bisection Method

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- Underlying Concept
  - Clearly, x_3 and x_1 can be used to compute yet another value.
  - This process is continued until $f(x) \approx 0$ or the desired accuracy is achieved.
  - It is to be noted that at each iteration, the new x value and one of the two previous values are used so that continuity and functional products are satisfied.

Bisection Method

■ Example 1: Bisection Method

Using graphical methods, the following function was found to have a real root between $x = 1$ and $x = 3$:

$$f(x) = x^3 - 5x^2 - 2x + 10$$

Approximate the root.

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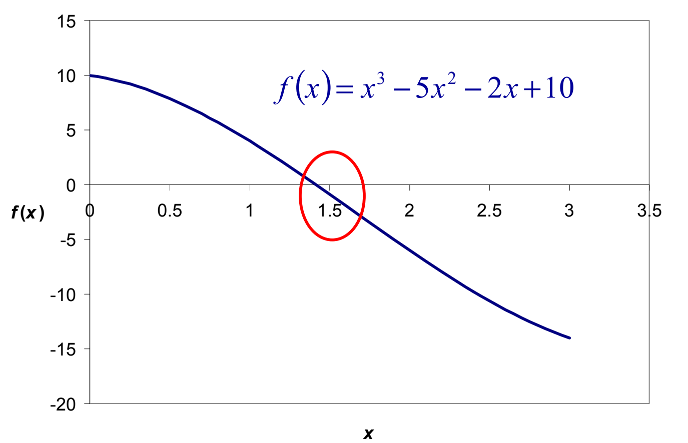
■ Example 1 (cont'd): Bisection Method



Bisection Method

- Example 1 (cont'd): Bisection Method

  Evaluating the function at the initial values:

  $x_1 = 1: \qquad f(x_1) = 4$

  $x_2 = 3: \qquad f(x_1) = -14$

  Obviously, $f(1)f(3) = (4)(-14) < 0$ and the root has a value between 1 and 3. Therefore, a new value is approximated by

  $$x_3 = \frac{1+3}{2} = 2: \qquad f(x_3) = f(2) = -6$$

Bisection Method

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- Example 1 (cont'd): Bisection Method

  It is evident that the root is between x_3 and x_1, which must now be used to compute a new x value. This is because $f(x_1)f(x_3) < 0$.

  Proceeding with the next five iterations, gives

  $$x_4 = \frac{1+2}{2} = 1.5: \qquad f(x_4) = f(1.5) = -0.875$$

  $$f(x_1)f(x_4) = f(1)f(1.5) = (4)(-0.875) < 0$$

Bisection Method

■ Example 1 (cont'd): Bisection Method

$$x_5 = \frac{x_1 + x_4}{2} = \frac{1 + 1.5}{2} = 1.25: \qquad f(x_5) = f(1.25) = 1.64063$$

$$f(x_1)f(x_5) = f(1)f(1.25) = (4)(1.875) > 0$$

$$f(x_4)f(x_5) = f(4)f(1.25) = (-0.875)(1.64063) < 0$$

$$x_6 = \frac{x_5 + x_4}{2} = \frac{1.25 + 1.5}{2} = 1.375: \qquad f(x_6) = f(1.375) = 0.39648$$

Bisection Method

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■ Example 1 (cont'd): Bisection Method

$$f(x_5)f(x_6) = f(1.25)f(1.375) = (1.64063)(0.39648) > 0$$

$$f(x_6)f(x_4) = f(1.375)f(1.5) = (0.39648)(-0.875) < 0$$

$$x_7 = \frac{x_6 + x_4}{2} = \frac{1.375 + 1.5}{2} = 1.4375: \qquad f(x_7) = f(1.4375) = -0.23657$$

$$f(x_6)f(x_7) = f(1.375)f(1.4375) = (0.39648)(-0.23657) < 0$$

Bisection Method

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■ Example 1 (cont'd): Bisection Method

$$x_7 = \frac{x_6 + x_7}{2} = \frac{1.375 + 1.4375}{2} = 1.40625: \qquad f(x_7) = f(1.40625) = 0.08072$$

It is evident that the functional values are approaching zero as the number of iterations is increased.

After six iteration the approximated root of 1.40625 compares favorably with the exact value of $\sqrt{2}$

Bisection Method

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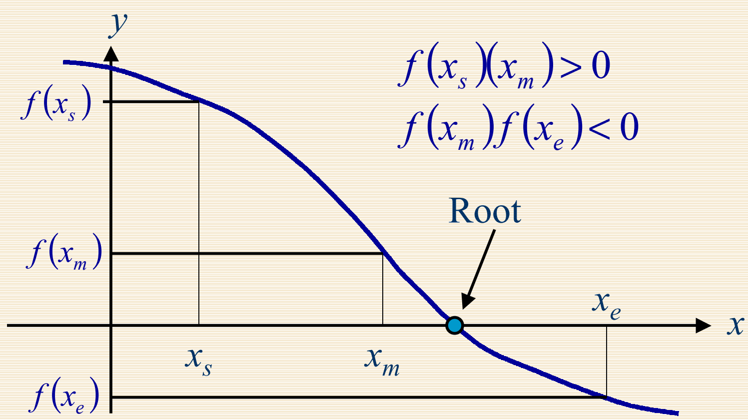
■ General Procedure

1. Sketch the function under consideration.
2. Establish the starting point x_s and the end point x_e of the interval such that $f(x_s)f(x_e) < 0$
3. From the starting point x_s and the end point x_e, locate the midpoint x_m at the center of the interval as follows:

$$x_m = \frac{x_s + x_e}{2}$$

Bisection Method

■ General Procedure



Bisection Method

■ General Procedure

4. At the starting point x_s, midpoint x_m, and the end point x_e, evaluate the function resulting from $f(x_s)$, $f(x_m)$, and $f(x_e)$, respectively.
5. Compute the product of the functions evaluated at the ends of the two intervals, that is, $f(x_s)f(x_m)$ and $f(x_m)f(x_e)$. The root lies in the interval for which the product is negative, and x_m is used as an estimate.

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- General Procedure
  5. Check for convergence as follows:
     a. If the convergence criterion (tolerance) is satisfied, then use x_m as the final estimate of the root.
     b. If the tolerance has not been met, specify the ends of half-interval in which the root is located as the starting and ending points for a new interval and go to step 3.

Bisection Method

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- Error Analysis and Convergence Criterion
  – To ensure closure of the iteration loop, a convergence criterion is needed to terminate the iterative procedure for finding the root of a function.
  – The convergence criterion used in step 6 of the bisection method can be expressed in terms of either the absolute value of the difference percent relative error.

Bisection Method

- Error Analysis and Convergence Criterion

$$\varepsilon_d = \left| x_{m,i+1} - x_{m,i} \right|$$

$$\varepsilon_r = \left| \frac{x_{m,i+1} - x_{m,i}}{x_{m,i+1}} \right| \times 100$$

Where ε_d = absolute difference, ε_r = percent relative error, $x_{m,i}$ = the midpoint in the previous root-search iteration, and $x_{m,i+1}$ is the midpoint in a new root-search iteration.

Bisection Method

- Error Analysis and Convergence Criterion
  - The true accuracy of the solution at any iteration can be computed if the the true solution (root x_t) is known. The true error ε_t in the i^{th} iteration is given by

$$\varepsilon_t = \left| \frac{x_t - x_{m,i}}{x_t} \right|$$

Bisection Method

■ Example 2: Bisection Method

The following polynomial has a root within the interval $3.75 \le x \le 5.00$:

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

If a tolerance of 0.01 (1%) is required, find this root using bisection method.

Bisection Method

■ Example 2: Bisection Method

$x_s = 3.75, \qquad x_e = 5.00 \qquad \boxed{f(x) = x^3 - x^2 - 10x - 8 = 0}$

$i = 1$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 5.00}{2} = 4.375$$

$$f(x_s) = f(3.75) = (3.75)^3 - (3.75)^2 - 10(3.75) - 8 = -6.828$$

$$f(x_m) = f(4.375) = (4.375)^3 - (4.375)^2 - 10(4.375) - 8 = 12.850$$

$$f(x_e) = f(5) = (5)^3 - (5)^2 - 10(5) - 8 = 42.000$$

$$f(x_s)f(x_m) < 0 \quad \text{(negative)}$$

$$f(x_m)f(x_e) > 0 \quad \text{(positive)}$$

Bisection Method

■ Example 2: Bisection Method

$$\boxed{f(x) = x^3 - x^2 - 10x - 8 = 0}$$

$$x_s = 3.75 \qquad x_e = 4.375$$

$$i = 2$$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 4.375}{2} = 4.063$$

$$f(x_s) = f(3.75) = -6.828$$

$$f(x_m) = f(4.063) = 1.918$$

$$f(x_e) = f(4.375) = 12.850$$

$f(x_s) f(x_m) < 0$ (negative)

Bisection Method

■ Example 2: Bisection Method

$$\text{error } \varepsilon_d = \left| x_{m,i+1} - x_{m,i} \right| = \left| 4.063 - 4.375 \right| = 0.312$$

$$\varepsilon_r = \left| \frac{x_{m,i+1} - x_{m,i}}{x_{m,i+1}} \right| \times 100 = \left| \frac{4.063 - 4.375}{4.063} \right| \times 100 = 7.68\%$$

$$x_s = 3.75 \qquad x_e = 4.063$$

$$i = 3$$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 4.063}{2} = 3.907$$

$$f(x_s) = f(3.75) = -6.828$$

$$f(x_m) = f(3.907) = -2.696$$

$$f(x_e) = f(4.063) = 1.934$$

$f(x_m) f(x_e) < 0$ (negative)

Bisection Method

■ Example 2: Bisection Method

Iteration i	x_s	x_m	x_e	$f(x_s)$	$f(x_m)$	$f(x_e)$	$f(x_s)f(x_m)$	$f(x_m)f(x_e)$	error ε_d	error ε_d
1	3.7500	4.3750	5.0000	-6.8281	12.8496	42.0000	-	+	---	---
2	3.7500	4.0625	4.3750	-6.8281	1.9182	12.8496	-	+	0.31250	7.69
3	3.7500	3.9063	4.0625	-6.8281	-2.7166	1.9182	+	-	0.15625	4.00
4	3.9063	3.9844	4.0625	-2.7166	-0.4661	1.9182	+	-	0.07813	1.96
5	3.9844	4.0234	4.0625	-0.4661	0.7092	1.9182	-	+	0.03906	0.97
6	3.9844	4.0039	4.0234	-0.4661	0.1174	0.7092	-	+	0.01953	0.49
7	3.9844	3.9941	4.0039	-0.4661	-0.1754	0.1174	+	-	0.00977	0.24
8	3.9941	3.9990	4.0039	-0.1754	-0.0293	0.1174	+	-	0.00488	0.12
9	3.9990	4.0015	4.0039	-0.0293	0.0440	0.1174	-	+	0.00244	0.06
10	3.9990	4.0002	4.0015	-0.0293	0.0073	0.0440	-	+	0.00122	0.03
11	3.9990	3.9996	4.0002	-0.0293	-0.0110	0.0073	+	-	0.00061	0.02
12	3.9996	3.9999	4.0002	-0.0110	-0.0018	0.0073	+	-	0.00031	0.01
13	3.9999	4.0001	4.0002	-0.0018	0.0027	0.0073	-	+	0.00015	0.00
14	3.9999	4.0000	4.0001	-0.0018	0.0005	0.0027	-	+	0.00008	0.00
15	3.9999	4.0000	4.0000	-0.0018	-0.0007	0.0005	+	-	0.00004	0.00

Bisection Method

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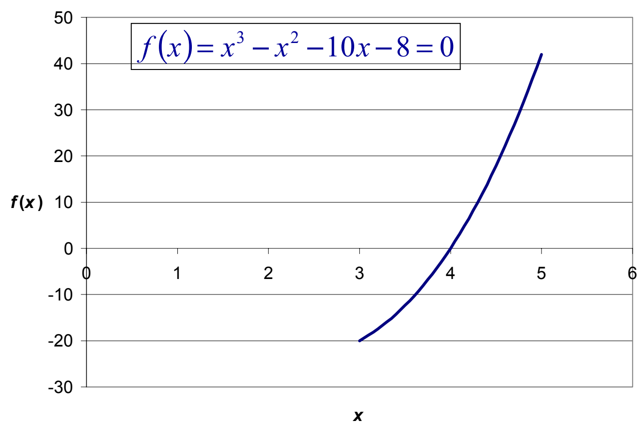
■ Example 2: Bisection Method



Bisection Method

- Disadvantage of Bisection Method
  - Although the bisection (interval halving) method will always converge on the root, the rate of convergence is very slow.
  - A faster method for converging on a single root of a function is the Newton-Raphson iteration Method.