CHAPTER 4c. ROOTS OF EQUATIONS

Bisection Method

- The bisection method or interval-halving is an extension of the direct-search method.
- It is used in cases where it is known that only one root occurs within a given interval of $x$.
- For the same level of precision, this method requires fewer calculations than the direct search method.
Bisection Method

■ Underlying Concept
  – The bisection (interval-halving) method is one of the simplest technique for determining roots of a function.
  – The basis for this method can be easily illustrated by considering the following function:

\[ y = f(x) \]

■ Underlying Concept
  – One objective is to find an \( x \) value for which \( y \) is zero.
  – Using this method, the function can be evaluated at two \( x \) values, say \( x_1 \) and \( x_2 \) such that

\[ f(x_1)f(x_2) < 0 \]
Bisection Method

- Underlying Concept
  - The implication is that one of the values is negative and the other is positive.
  - Also, the function must be continuous for
    \[ x_1 \leq x \leq x_2 \]
  - These conditions can be easily satisfied by sketching the function as shown in the following figure:
Bisection Method

Underlying Concept

- Looking at the figure, it is clear that the function is negative at $x_1$ and positive at $x_2$, and is continuous for $x_1 \leq x \leq x_2$.
- Therefore, the root must between $x_1$ and $x_2$ and a new approximation to the root can be calculated as

\[ x_3 = \frac{x_1 + x_2}{2} \]

- Clearly, $x_3$ and $x_1$ can be used to compute yet another value.
- This process is continued until $f(x) \approx 0$ or the desired accuracy is achieved.
- It is to be noted that at each iteration, the new $x$ value and one of the two previous values are used so that continuity and functional products are satisfied.
Example 1: Bisection Method

Using graphical methods, the following function was found to have a real root between \( x = 1 \) and \( x = 3 \):

\[
f(x) = x^3 - 5x^2 - 2x + 10
\]

Approximate the root.
Example 1 (cont’d): Bisection Method

Evaluating the function at the initial values:

\[ f(x_1) = 4 \]
\[ f(x_2) = -14 \]

Obviously, \( f(1)f(3) = (4)(-14) < 0 \) and the root has a value between 1 and 3. Therefore, a new value is approximated by

\[ x_3 = \frac{1+3}{2} = 2 \]
\[ f(x_3) = f(2) = -6 \]

Proceeding with the next five iterations, gives

\[ x_4 = \frac{1+2}{2} = 1.5 \]
\[ f(x_4) = f(1.5) = -0.875 \]
\[ f(x_1)f(x_4) = f(1)f(1.5) = (4)(-0.875) < 0 \]
Bisection Method

Example 1 (cont’d): Bisection Method

\[ x_5 = \frac{x_3 + x_4}{2} = \frac{1+1.5}{2} = 1.25 : \quad f(x_5) = f(1.25) = 1.64063 \]
\[ f(x_3)f(x_5) = f(1)f(1.25) = (4)(1.875) > 0 \]
\[ f(x_4)f(x_5) = f(4)f(1.25) = (-0.875)(1.64063) < 0 \]

\[ x_6 = \frac{x_5 + x_4}{2} = \frac{1.25+1.5}{2} = 1.375 : \quad f(x_6) = f(1.375) = 0.39648 \]

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Example 1 (cont’d): Bisection Method

\[ f(x_5)f(x_6) = f(1.25)f(1.375) = (1.64063)(0.39648) > 0 \]
\[ f(x_6)f(x_4) = f(1.375)f(1.5) = (0.39648)(-0.875) < 0 \]
\[ x_7 = \frac{x_6 + x_4}{2} = \frac{1.375+1.5}{2} = 1.4375 : \quad f(x_7) = f(1.4375) = -0.23657 \]
\[ f(x_6)f(x_7) = f(1.375)f(1.4375) = (0.39648)(-0.23657) < 0 \]
Example 1 (cont’d): Bisection Method

After six iteration the approximated root of 1.40625 compares favorably with the exact value of $\sqrt{2}$.

General Procedure

1. Sketch the function under consideration.
2. Establish the starting point $x_s$ and the end point $x_e$ of the interval such that $f(x_s)f(x_e) < 0$.
3. From the starting point $x_s$ and the end point $x_e$, locate the midpoint $x_m$ at the center of the interval as follows:

$$x_m = \frac{x_s + x_e}{2}$$
Bisection Method

General Procedure

4. At the starting point $x_s$, midpoint $x_m$, and the end point $x_e$, evaluate the function resulting from $f(x_s)$, $f(x_m)$, and $f(x_e)$, respectively.

5. Compute the product of the functions evaluated at the ends of the two intervals, that is, $f(x_s)f(x_m)$ and $f(x_m)f(x_e)$. The root lies in the interval for which the product is negative, and $x_m$ is used as an estimate.
Bisection Method

General Procedure

5. Check for convergence as follows:
   a. If the convergence criterion (tolerance) is satisfied, then use $x_{mf}$ as the final estimate of the root.
   b. If the tolerance has not been met, specify the ends of half-interval in which the root is located as the starting and ending points for a new interval and go to step 3.

Error Analysis and Convergence Criterion

– To ensure closure of the iteration loop, a convergence criterion is needed to terminate the iterative procedure for finding the root of a function.
– The convergence criterion used in step 6 of the bisection method can be expressed in terms of either the absolute value of the difference percent relative error.
Bisection Method

- Error Analysis and Convergence Criterion
  \[ \varepsilon_d = |x_{m,i+1} - x_{m,i}| \]
  \[ \varepsilon_r = \left| \frac{x_{m,i+1} - x_{m,i}}{x_{m,i+1}} \right| \times 100 \]

Where \( \varepsilon_d \) = absolute difference, \( \varepsilon_r \) = percent relative error, \( x_{m,i} \) = the midpoint in the previous root-search iteration, and \( x_{m,i+1} \) is the midpoint in a new root-search iteration.

Bisection Method

- Error Analysis and Convergence Criterion
  - The true accuracy of the solution at any iteration can be computed if the true solution (root \( x_t \)) is known. The true error \( \varepsilon_t \) in the \( i \)th iteration is given by
  \[ \varepsilon_t = \left| \frac{x_t - x_{m,i}}{x_t} \right| \]
### Example 2: Bisection Method

The following polynomial has a root within the interval $3.75 \leq x \leq 5.00$:

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

If a tolerance of 0.01 (1%) is required, find this root using bisection method.

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**Step 1:**

- $x_s = 3.75$, $x_e = 5.00$
- $i = 1$

**Step 2:**

- $x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 5.00}{2} = 4.375$

**Step 3:**

- $f(x_s) = f(3.75) = (3.75)^3 - (3.75)^2 - 10(3.75) - 8 = -6.828$
- $f(x_e) = f(5) = (5)^3 - (5)^2 - 10(5) - 8 = 42.000$
- $f(x_m) = f(4.375) = (4.375)^3 - (4.375)^2 - 10(4.375) - 8 = 12.850$

**Step 4:**

- $f(x_s)f(x_e) < 0$ (negative)
- $f(x_s)f(x_m) > 0$ (positive)
Bisection Method

Example 2: Bisection Method

\[ f(x) = x^3 - x^2 - 10x - 8 = 0 \]

\[ x_l = 3.75 \quad x_u = 4.375 \]

\[ i = 2 \]

\[ x_m = \frac{x_l + x_u}{2} = \frac{3.75 + 4.375}{2} = 4.063 \]

\[ f(x_l) = f(3.75) = -6.828 \]

\[ f(x_u) = f(4.063) = 1.918 \]

\[ f(x_m) = f(4.063) = 12.850 \]

\[ f(x_l)f(x_m) < 0 \text{ (negative) } \]

\[ f(x_u)f(x_m) < 0 \text{ (negative) } \]

Example 2: Bisection Method

error \( \varepsilon \) = \[ |x_{m,i+1} - x_{m,i}| = |4.063 - 4.375| = 0.312 \]

\[ \varepsilon = \left| \frac{x_{m,i+1} - x_{m,i}}{x_{m,i+1}} \right| \times 100 = \frac{4.063 - 4.375}{4.063} \times 100 = 7.68\% \]

\[ x_l = 3.75 \quad x_u = 4.063 \]

\[ i = 3 \]

\[ x_m = \frac{x_l + x_u}{2} = \frac{3.75 + 4.063}{2} = 3.907 \]

\[ f(x_l) = f(3.75) = -6.828 \]

\[ f(x_u) = f(3.907) = -2.696 \]

\[ f(x_m) = f(4.063) = 1.934 \]

\[ f(x_m)f(x_u) < 0 \text{ (negative) } \]

\[ f(x_u)f(x_l) < 0 \text{ (negative) } \]
Bisection Method

Example 2: Bisection Method

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<th>Iteration</th>
<th>$x_s$</th>
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<th>$x_e$</th>
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\[ f(x) = x^3 - x^2 - 10x - 8 = 0 \]
Disadvantage of Bisection Method

- Although the bisection (interval halving) method will always converge on the root, the rate of convergence is very slow.
- A faster method for converging on a single root of a function is the Newton-Raphson iteration Method.