CHAPTER 4a.
ROOTS OF EQUATIONS

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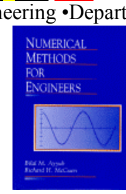

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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Introduction

- Quadratic Formula

In high school, students usually learn how to use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

to solve a quadratic equation of the type

$$f(x) = ax^2 + bx + c = 0 \tag{2}$$

Introduction

- Quadratic Formula
  - The values calculated with Eq. 1 are called the "*roots*" of Eq.1.
  - They represent the values of *x* that make Eq. 1 equal to zero.
  - Thus , the root (roots) of an equation can be defined as the value (values) of *x* that makes the equation equal to zero.

Introduction

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- Root of Equation
  - Definition

    The root (roots) of an equation is defined as the value (values) of *x* that make the equation

    $$f(x) = 0$$

    equal to zero. The roots of an equation sometimes are called the *zeros* of the equation.

Introduction

- Analytical Solution
  - The roots (two roots) for a quadratic equation are said to be found analytically.
  - The are found through the use of the quadratic formula.
  - For third-order polynomial, the roots (three roots) can also be found analytically. However, no general solution exists for other higher-order polynomials.

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- Numerical Solution
  - The solution of many scientific and engineering problems requires finding the roots of equations that are complex and nonlinear in nature.
  - For example, the function $f(x) = e^{-x} - x$ cannot be solved analytically.
  - In such instances, the only alternative is an approximation by numerical methods.

ENCE 203 – CHAPTER 4a. ROOTS OF EQUATIONS
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Introduction

- Numerical Solution
  - To obtain the roots of $f(x) = e^{-x} - x = 0$, some type of iterative numerical method must be employed.
  - This is, in general, requires a large number of calculations, particularly, if the roots are to be determined to high degree precision.
  - Thus the problem is well suited to numerical analysis.

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- Types of Equations
  - There are typically two types of equations that relate to roots finding:
    - Algebraic
    - Transcendental Equations
  - *Polynomials* are a simple class of algebraic functions that represented generally by

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Introduction

- Types of Equations
  - Some specific examples of algebraic (polynomials) equations are

$$f_2(x) = 2 - 3.47x + 8.5x^2$$

and

$$f_2(x) = 4x^2 - x^3 + 6x^6$$

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- Types of Equations
  - A *transcendental* function is one that is non- algebraic.
  - These include trigonometric, exponential, logarithmic, and others.
  - Specific examples are

$$f(x) = \left(\frac{x}{2}\right)^2 - \sin(x) = 0$$

and

$$f(x) = \ln(x^2) - 1$$

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■ Roots of Equations

– Depending on the type, an equation can have one, two, or more roots.

– Furthermore, the roots of equations can be either real or complex. Example complex roots are $x_1 = 1+2i$ and $x_2 = 1\text{-}2i$ of the following quadratic equation:

$$x^2 - 2x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

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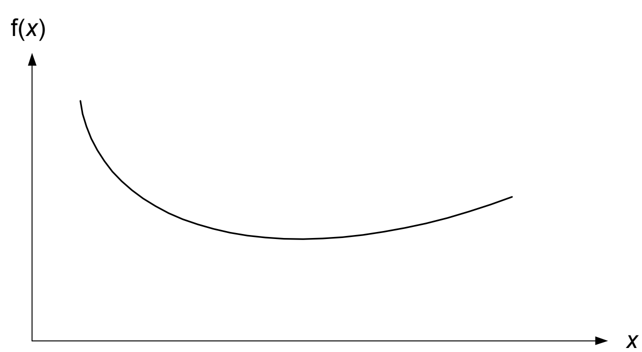
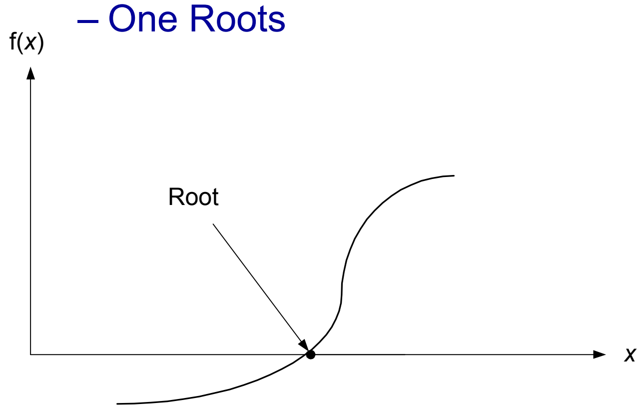
■ Roots of Equations

– No Roots



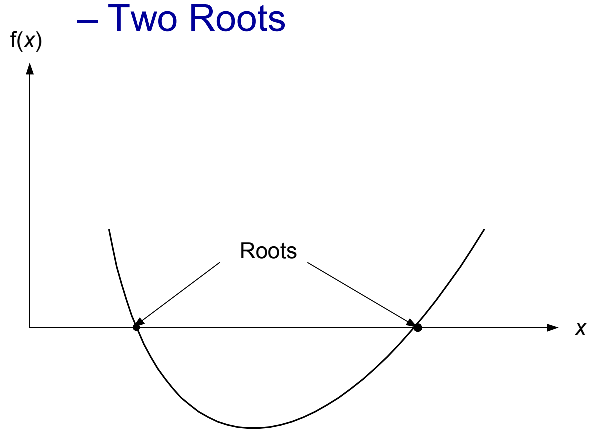
Introduction

- Roots of Equations
  - One Roots



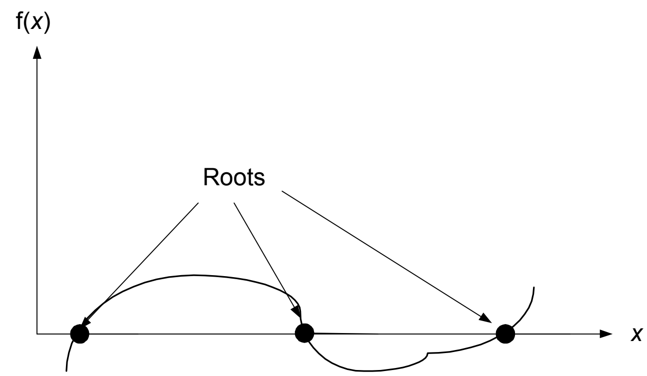
Introduction

- Roots of Equations
  - Two Roots



Introduction

- Roots of Equations
  - Three Roots



Introduction

- Roots of Equations
  - Although there are situations where complex roots of nonpolynomials are of interest, such cases are less common than for polynomials.
  - The standard methods for locating roots typically fall into two somewhat related but primarily distinct classes of problems:

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- Roots of Equations
  1. *The determination of the real roots of algebraic and transcendental equations*. These techniques are usually designed to determine the value of a single real root on the basis of its approximate location.
  2. *The determination of all real and complex roots of polynomials*. These methods are specifically designed for polynomials. The systematically determine all roots of the polynomials rather than determining a single real root given an approximate location.

Eigenvalue Analysis

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- Engineering Applications
  – A large number of engineering problems require the determination of a set of values called *eigenvalues* or *characteristic values*.
  – The electrical engineer, for example, uses eigenvalue analysis in the solution of two-terminal networks and in the optimization of adjustments of a control system.

Eigenvalue Analysis

- Engineering Applications
  - A structural engineer uses eigenvalue analysis in the design of a structure to resist ground motion due an earthquake.
  - The chemical engineer uses eigenvalue analysis in the design of reactor systems.
  - The aeronautical engineer applies eigenvalue analyses in analyzing the flutter of an airplane wing.

Eigenvalue Analysis

- What are Egienvalues?

  Eigenvalues or characteristic values are values, usually denoted as λ, for which the followibg matrix system has a nonzero (I.e., nontrivial) solution ***X***.

$$[A-\lambda I]X = 0 \qquad (3)$$

$A = n \times n$ matrix

$I = n \times n$ identity diagonal matrix

$\lambda =$ parameter called eigen value

Eigenvalue Analysis

- Eigenvalues Characteristic Equation

  The characteristic equations of the eigenvalues can be obtained by expanding the following expression:

$$\left| A - \lambda I \right| = 0 \qquad (4)$$

  into a polynomial, and then set this polynomial equal to zero.

  Solving this equation for the roots λ's gives the eigenvalues.

Eigenvalue Analysis

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- Example 1: 3×3 matrix:

  Find the eigenvalues of the following matrix matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Eigenvalue Analysis

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■ Example 1 (cont'd): 3 × 3 matrix:

$$A-\lambda I=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}-\lambda&a_{12}&a_{13}\\a_{21}&a_{22}-\lambda&a_{23}\\a_{31}&a_{32}&a_{33}-\lambda\end{bmatrix}$$

Eigenvalue Analysis

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■ Example 1 (cont'd): 3 × 3 matrix:

$$|A-\lambda I|=\begin{vmatrix}a_{11}-\lambda&a_{12}&a_{13}\\a_{21}&a_{22}-\lambda&a_{23}\\a_{31}&a_{32}&a_{33}-\lambda\end{vmatrix}=0$$

or

$$(a_{11}-\lambda)[(a_{22}-\lambda)(a_{33}-\lambda)-a_{23}a_{32}]-a_{12}[a_{21}(a_{33}-\lambda)-a_{23}a_{31}]+a_{13}[a_{21}a_{32}-(a_{22}-\lambda)a_{31}]=0$$

or

$$\lambda^3+b_2\lambda^2+b_1\lambda+b_0=0$$

where b_0, b_1, and b_2 are functions of the elements a_{ij} of $\boldsymbol{A}$. The solution (roots) of the characteristic equation provides the three eigenvalues.

Eigenvalue Analysis

- Example 2: 2×2 Matrix

  Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

Eigenvalue Analysis

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- Example 2 (cont'd): 2×2 Matrix

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 2(4)$$

$$= \lambda^2 - 4\lambda - 5$$

The characteristic equation of ***A*** is det(***A*** - λ), or

$$\lambda^2 - 4\lambda - 5 = 0$$

Solving for λ, we get λ_1 = -1, and λ_2 = 5
Hence, the eigenvalues of ***A*** are -1 and 5.

Eigenvalue Analysis

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■ Example 3: 2 × 2 Matrix

– Find the eigenvalues of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}$$

Eigenvalue Analysis

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■ Example 3 (cont'd): 2 × 2 Matrix

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - (-2)(1)$$

$$= \lambda^2 - 2\lambda + 3$$

The characteristic equation of ***A*** is det(***A*** - λ), or

$$\lambda^2 - 2\lambda + 3 = 0$$

Solving for λ, we get $\lambda_1 = 1 + \sqrt{2}i$, and $\lambda_2 = 1 - \sqrt{2}i$

Hence, the eigenvalues of ***A*** are $1 + \sqrt{2}i$, and $1 - \sqrt{2}i$

<u>Note:</u> Even if the elements of ***A*** are real, the eigenvalues may be complex.

Methods for Finding the Roots of Equations

1. Graphical Methods
2. Direct-Search Method
3. Bisection Method
4. Newton-Raphson Iteration
5. Secant Method
6. Polynomial Reduction and Synthetic Division

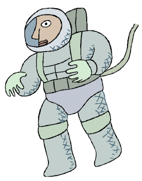
Graphical Methods

- One method to obtain an approximate solution is to plot the function and determine where it crosses the *x* axis. This point, which represents the *x* value for which $f(x) = 0$, is the root.
- Although graphical methods are useful for obtaining rough estimates of roots, they are limited due to their lack of precision.

Graphical Methods

- Example: Falling Parachutist Problem

  Using the graphical approach to determine the drag coefficient for a parachutist of mass m = 68.1 kg to have a velocity of 40 m/s after free falling for t = 10 s. Note that the acceleration due to gravity 9.8 m/s^2.



$$\boxed{v(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]} \qquad (4)$$

Graphical Methods

- Example: Falling Parachutist Problem

  Equation 4 can be rewritten as

$$f(c) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right] - v = 0$$

or

$$f(c) = \frac{9.8(68.1)}{c}\left(1 - e^{-(c/68.1)(10)}\right) - 40 = 0$$

or

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40 = 0$$

Graphical Methods

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- Example (cont'd): Falling Parachutist Problem
  - Various values of *c* can be substituted into the right-hand side of

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40$$

  - To check approximately which one will make the function $f(x) = 0$.
  - The following table and plot show the results.

Graphical Methods

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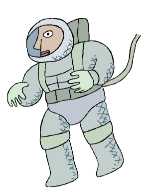
- Example (cont'd): Falling Parachutist Problem

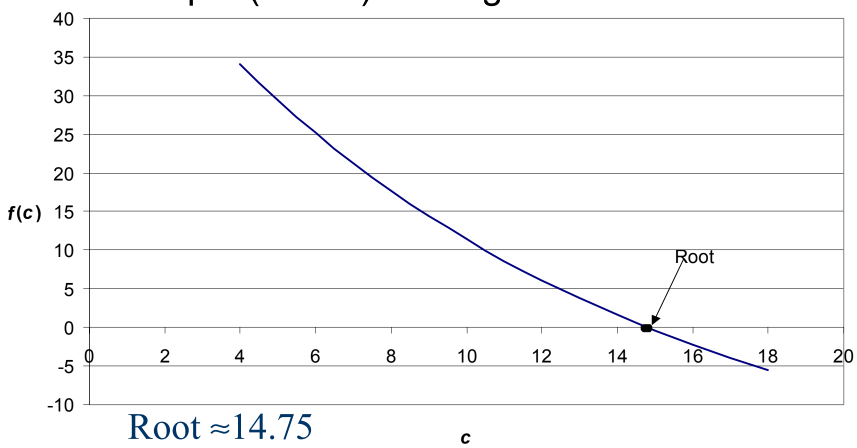
$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40$$



c	*f*(*c*)	*c*	*f*(*c*)
4	34.115	12	6.0669
5	29.423	13	3.727
6	25.142	14	1.5687
7	21.231	15	-0.425
8	17.653	16	-2.269
9	14.376	17	-3.977
10	11.369	18	-5.561
11	8.6073	19	-7.032

Graphical Methods

■ Example (cont'd): Falling Parachutist Problem



Root ≈ 14.75

Graphical Methods

■ Example: Eigenvalues

The following characteristic equation resulted from the matrix ***A***:

$$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188$$

$$A = \begin{bmatrix} 1 & -0.42 & -0.61 \\ -0.42 & 1 & 0.37 \\ -0.61 & 0.37 & 1 \end{bmatrix}$$

Estimate the eigenvalues (roots) by graphical approach.

Graphical Methods

■ Example (cont'd): Eigenvalues

λ	$f(\lambda)$	λ	$f(\lambda)$	λ	$f(\lambda)$	λ	$f(\lambda)$
0.00	-0.50419	0.55	0.027717	1.10	-0.25713	1.65	-0.36047
0.05	-0.39583	0.60	0.020572	1.15	-0.28902	1.70	-0.32637
0.10	-0.30173	0.65	0.007427	1.20	-0.31867	1.75	-0.28176
0.15	-0.22112	0.70	-0.01097	1.25	-0.34531	1.80	-0.22591
0.20	-0.15327	0.75	-0.03386	1.30	-0.36821	1.85	-0.15805
0.25	-0.09741	0.80	-0.06051	1.35	-0.3866	1.90	-0.07745
0.30	-0.05281	0.85	-0.09015	1.40	-0.39975	1.95	0.016657
0.35	-0.0187	0.90	-0.12205	1.45	-0.40689	2.00	0.125012
0.40	0.005652	0.95	-0.15544	1.50	-0.40729	2.05	0.248367
0.45	0.021007	1.00	-0.18959	1.55	-0.40018	2.10	0.387472
0.50	0.028112	1.05	-0.22373	1.60	-0.38483	2.15	0.543077

$$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188$$

Graphical Methods

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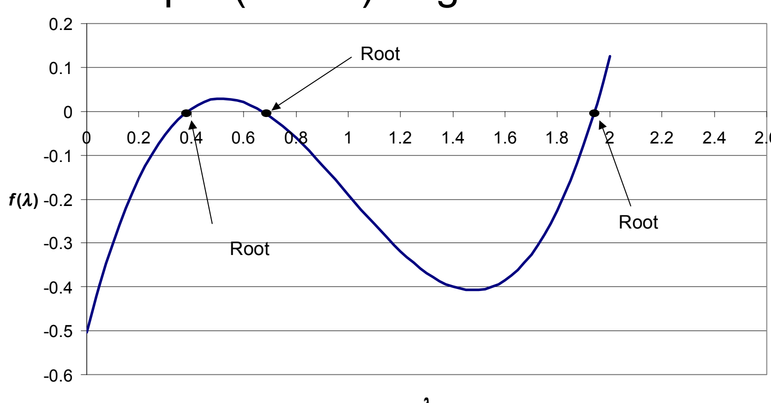
■ Example (cont'd): Eigenvalues



$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188$

$\lambda_1 \approx 0.4$, $\lambda_2 \approx 0.7$, and $\lambda_3 \approx 1.9$