CHAPTER 4a.
ROOTS OF EQUATIONS

Introduction

Quadratic Formula

In high school, students usually learn how to use the quadratic formula

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]  \hspace{1cm} (1)

...
Introduction

■ Quadratic Formula
  – The values calculated with Eq. 1 are called the “roots” of Eq. 1.
  – They represent the values of \( x \) that make Eq. 1 equal to zero.
  – Thus, the root (roots) of an equation can be defined as the value (values) of \( x \) that makes the equation equal to zero.

Introduction

■ Root of Equation
  – Definition
  The root (roots) of an equation is defined as the value (values) of \( x \) that make the equation

\[
f(x) = 0
\]

equal to zero. The roots of an equation sometimes are called the zeros of the equation.
Introduction

- **Analytical Solution**
  - The roots (two roots) for a quadratic equation are said to be found analytically.
  - They are found through the use of the quadratic formula.
  - For third-order polynomial, the roots (three roots) can also be found analytically. However, no general solution exists for other higher-order polynomials.

- **Numerical Solution**
  - The solution of many scientific and engineering problems requires finding the roots of equations that are complex and nonlinear in nature.
  - For example, the function $f(x) = e^{-x} - x$ cannot be solved analytically.
  - In such instances, the only alternative is an approximation by numerical methods.
**Introduction**

- **Numerical Solution**
  - To obtain the roots of $f(x) = e^{-x} - x = 0$, some type of iterative numerical method must be employed.
  - This is, in general, requires a large number of calculations, particularly, if the roots are to be determined to high degree precision.
  - Thus the problem is well suited to numerical analysis.

- **Types of Equations**
  - There are typically two types of equations that relate to roots finding:
    - **Algebraic**
    - **Transcendental Equations**
  - *Polynomials* are a simple class of algebraic functions that represented generally by
    $$f_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$
Introduction

Types of Equations

- Some specific examples of algebraic (polynomials) equations are

\[ f_2(x) = 2 - 3.47x + 8.5x^2 \]

and

\[ f_2(x) = 4x^2 - x^3 + 6x^6 \]

- A transcendental function is one that is non-algebraic.

- These include trigonometric, exponential, logarithmic, and others.

- Specific examples are

\[ f(x) = \left(\frac{x}{2}\right)^2 - \sin(x) = 0 \]

and

\[ f(x) = \ln(x^2) - 1 \]
Introduction

Roots of Equations

– Depending on the type, an equation can have one, two, or more roots.

– Furthermore, the roots of equations can be either real or complex. Example complex roots are $x_1 = 1 + 2i$ and $x_2 = 1 - 2i$ of the following quadratic equation:

$$x^2 - 2x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

Introduction

Roots of Equations

– No Roots

$$f(x)$$

$x$
Introduction

- Roots of Equations
  - One Roots

\[ f(x) \]

\[ x \]

- Roots of Equations
  - Two Roots

\[ f(x) \]

\[ x \]
Introduction

Roots of Equations

- Three Roots

Although there are situations where complex roots of nonpolynomials are of interest, such cases are less common than for polynomials.

- The standard methods for locating roots typically fall into two somewhat related but primarily distinct classes of problems:
Introduction

Roots of Equations

1. The determination of the real roots of algebraic and transcendental equations. These techniques are usually designed to determine the value of a single real root on the basis of its approximate location.

2. The determination of all real and complex roots of polynomials. These methods are specifically designed for polynomials. The systematically determine all roots of the polynomials rather than determining a single real root given an approximate location.

Eigenvalue Analysis

Engineering Applications

- A large number of engineering problems require the determination of a set of values called eigenvalues or characteristic values.
- The electrical engineer, for example, uses eigenvalue analysis in the solution of two-terminal networks and in the optimization of adjustments of a control system.
Engineering Applications

- A structural engineer uses eigenvalue analysis in the design of a structure to resist ground motion due an earthquake.
- The chemical engineer uses eigenvalue analysis in the design of reactor systems.
- The aeronautical engineer applies eigenvalue analyses in analyzing the flutter of an airplane wing.

Eigenvalue Analysis

What are Eigenvalues?

Eigenvalues or characteristic values are values, usually denoted as $\lambda$, for which the following matrix system has a nonzero (i.e., nontrivial) solution $X$.

$$[A - \lambda I]X = 0 \quad (3)$$

- $A = n \times n$ matrix
- $I = n \times n$ identity diagonal matrix
- $\lambda = \text{parameter called eigen value}$
Eigenvalue Analysis

■ Eigenvalues Characteristic Equation

The characteristic equations of the eigenvalues can be obtained by expanding the following expression:

\[ \begin{vmatrix} A - \lambda I \end{vmatrix} = 0 \quad (4) \]

into a polynomial, and then set this polynomial equal to zero. Solving this equation for the roots \( \lambda \)'s gives the eigenvalues.

Example 1: 3 \( \times \) 3 matrix:

Find the eigenvalues of the following matrix matrix:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]
Eigenvalue Analysis

Example 1 (cont’d): 3 × 3 matrix:

\[
A - \lambda I = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} - \lambda
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} - \lambda
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a_{11} - \lambda & a_{12} & a_{13} \\
a_{21} & a_{22} - \lambda & a_{23} \\
a_{31} & a_{32} & a_{33} - \lambda
\end{bmatrix}
\]

\[
\begin{vmatrix}
a_{11} - \lambda & a_{12} & a_{13} \\
a_{21} & a_{22} - \lambda & a_{23} \\
a_{31} & a_{32} & a_{33} - \lambda
\end{vmatrix} = 0
\]

or

\[
(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) - a_{23}a_{32} - a_{12}a_{31}(a_{23} - \lambda) - a_{23}a_{11}(a_{22} - \lambda) + a_{11}a_{22}a_{33} - (a_{22} - \lambda)a_{31} = 0
\]

or

\[
\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0
\]

where \(b_0, b_1,\) and \(b_2\) are functions of the elements \(a_{ij}\) of \(A\). The solution (roots) of the characteristic equation provides the three eigenvalues.
Example 2: 2 × 2 Matrix

Find the eigenvalues of

\[ A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \]

\[ A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{bmatrix} \]

\[ \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{vmatrix} = (1 - \lambda)(3 - \lambda) - 2(4) \]

\[ = \lambda^2 - 4\lambda - 5 \]

The characteristic equation of \( A \) is \( \det(A - \lambda) \), or

\[ \lambda^2 - 4\lambda - 5 = 0 \]

Solving for \( \lambda \), we get \( \lambda_1 = -1 \), and \( \lambda_2 = 5 \)

Hence, the eigenvalues of \( A \) are -1 and 5.
Example 3: 2 × 2 Matrix

- Find the eigenvalues of

\[ A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \]

\[ A - \lambda I = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & -2 \\ 1 & 1 - \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & -2 \\ 1 & 1 - \lambda \end{bmatrix} \]

The characteristic equation of \( A \) is det\((A - \lambda I)\), or

\[ \lambda^2 - 2\lambda + 3 = 0 \]

Solving for \( \lambda \), we get \( \lambda_1 = 1 + \sqrt{2}i \) and \( \lambda_2 = 1 - \sqrt{2}i \). Hence, the eigenvalues of \( A \) are \( 1 + \sqrt{2}i \) and \( 1 - \sqrt{2}i \).

Note: Even if the elements of \( A \) are real, the eigenvalues may be complex.
Methods for Finding the Roots of Equations

1. Graphical Methods
2. Direct-Search Method
3. Bisection Method
4. Newton-Raphson Iteration
5. Secant Method
6. Polynomial Reduction and Synthetic Division

Graphical Methods

- One method to obtain an approximate solution is to plot the function and determine where it crosses the x axis. This point, which represents the x value for which $f(x) = 0$, is the root.

- Although graphical methods are useful for obtaining rough estimates of roots, they are limited due to their lack of precision.
Graphical Methods

- Example: Falling Parachutist Problem

Using the graphical approach to determine the drag coefficient for a parachutist of mass \( m = 68.1 \text{ kg} \) to have a velocity of 40 m/s after free falling for \( t = 10 \text{ s} \). Note that the acceleration due to gravity 9.8 m/s\(^2\).

\[
\nu(t) = \frac{gm}{c} \left[ 1 - e^{- \frac{c}{m} t} \right]
\]  

(4)

Graphical Methods

- Example: Falling Parachutist Problem

Equation 4 can be rewritten as

\[
f(c) = \frac{gm}{c} \left[ 1 - e^{- \frac{c}{m} t} \right] - \nu = 0
\]

or

\[
f(c) = \frac{9.8(68.1)}{c} \left( 1 - e^{- \frac{c}{68.1} (10)} \right) - 40 = 0
\]

or

\[
f(c) = \frac{667.38}{c} \left( 1 - e^{-0.146843c} \right) - 40 = 0
\]
Graphical Methods

Example (cont’d): Falling Parachutist Problem

– Various values of $c$ can be substituted into the right-hand side of

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40$$

– To check approximately which one will make the function $f(x) = 0$.
– The following table and plot show the results.

\[
\begin{array}{ccc}
\hline
\text{c} & \text{f(c)} & \text{c} & \text{f(c)} \\
\hline
4 & 34.115 & 12 & 6.0669 \\
5 & 29.423 & 13 & 3.727 \\
6 & 25.142 & 14 & 1.5687 \\
7 & 21.231 & 15 & -0.425 \\
8 & 17.653 & 16 & -2.269 \\
9 & 14.376 & 17 & -3.977 \\
10 & 11.369 & 18 & -5.561 \\
11 & 8.6073 & 19 & -7.032 \\
\hline
\end{array}
\]
Graphical Methods

**Example (cont’d): Falling Parachutist Problem**

-10
-5
0
5
10
15
20
25
30
35
40

f(c)

Root ≈ 14.75

Graphical Methods

**Example: Eigenvalues**

The following characteristic equation resulted from the matrix $A$:

\[ f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188 \]

\[
A = \begin{bmatrix}
1 & -0.42 & -0.61 \\
-0.42 & 1 & 0.37 \\
-0.61 & 0.37 & 1
\end{bmatrix}
\]

Estimate the eigenvalues (roots) by graphical approach.
Graphical Methods

Example (cont’d): Eigenvalues

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( f(\lambda) )</th>
<th>( \lambda )</th>
<th>( f(\lambda) )</th>
<th>( \lambda )</th>
<th>( f(\lambda) )</th>
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<td>-0.50419</td>
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<td>-0.01092</td>
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<td>0.95</td>
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<td>-0.22373</td>
<td>1.60</td>
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</tr>
</tbody>
</table>

\[
f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188
\]