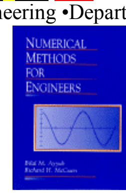
CHAPTER 2d. MATRICES



by

Dr. Ibrahim A. Assakkaf

Spring 2001

ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

University of Maryland, College Park

Determinants

- Every square matrix has associated with it a scalar called *determinant*.
- There are different methods to find the determinant of a square matrix.
- Among these methods, is the widely used method of finding the determinant with expansion by cofactors.

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- Notation
  - Given a square matrix ***A***, the determinant of this matrix is denoted by either

    det(***A***) or |***A***|
  - For example if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{then} \quad \det(A) = |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

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- Notation

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \det(A) = |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

  - Note that ***A*** represents a matrix, a rectangular array, an entity unto itself, while det(***A***) represents a scalar, a number associated with the matrix ***A.***
  - The difference is only in the form.

Determinants

■ Determinant of 1 × 1 matrix

– The determinant of 1 × 1 matrix ***A*** = [a] is the scalar a.

– Example

• The determinant of the matrix [5] is 5 and the determinant of the matrix [-0.23] is –0.23

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■ Determinant of 2 × 2 matrix

By definition, the determinant of a 2 × 2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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- Example: 2×2 matrix

  Find det(***A***) if

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = ad - bc = (1)(3) - (2)(4)$$

$$= 3 - 8 = -5$$

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- Method for Finding the Determinant of Higher-order Matrices
  - Expansion by Cofactor
  - <u>Definition:</u>

    "Given a matrix ***A***, a *<u>minor</u>* is the determinant of any square submatrix of ***A***"

    That is, given a square matrix ***A***, a minor is the determinant formed by ***A*** by removal of an equal number of rows and columns

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■ Expansion by Cofactor

– Examples: Minors

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

then $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$ and $\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$

are both minors since

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$\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$ and $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$

are both square submatrices of ***A***, while

$\begin{vmatrix} 1 & 2 \\ 8 & 9 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 \end{vmatrix}$

are not minors since

$$\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$$

is not a submatrix of ***A*** and [1 2], although a submatrix of ***A***, is not square.

Determinants

■ Expansion by Cofactor

– Definition:

"Given a matrix ***A*** = [a_{ij}], the *cofactor* of the element a_{ij} is a scalar obtained by multiplying together the term $(-1)^{i+j}$ and the minor obtained from ***A*** by removing the i^{th} row and j^{th} column"

In other words, to compute the cofactor of the element a_{ij} we first form a submatrix of ***A*** by crossing out both the row and column in which the element a_{ij} appears. Then we find the determinant of the submatrix and finally multiply it by the number $(-1)^{i+j}$

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■ Example 1: Cofactor

– Find the cofactor of the element 4 in the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We first note that 4 appears in the (2,1) position. The submatrix obtained by crossing out the second row and first column is

Determinants

■ Example 1 (cont'd): Cofactor

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$$

$$\det\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 2(9) - 3(8) = -6$$

Since 4 appears in the (2,1) position, $i = 2$, and $j = 1$.
Thus, $(-1)^{i+j} = (-1)^{2+1} = (-1)^3 = -1$
Therefore, the cofactor of 4
$= (-1) \times (-6) = 6$

Determinants

■ Example 2: Cofactor

– Find the cofactor of the element 9 in the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We first note that 9 appears in the (3,3) position.
The submatrix obtained by crossing out the third row and third column is

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■ Example 2 (cont'd): Cofactor

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 1(5) - 2(4) = -3$$

Since 9 appears in the (3,3) position, $i = 3$, and $j = 3$.
Thus, $(-1)^{i+j} = (-1)^{3+3} = (-1)^6 = 1$
Therefore, the cofactor of 9

$= (1) \times (-3) = -3$

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■ Expansion by Cofactors

– To find the determinant of a square matrix ***A*** of arbitrary order:

1. Pick any one row or any one column of the matrix.
2. For each element in the row or column selected, find its cofactor.
3. Multiply each element in the row or column selected by its cofactor and sum the results.
4. This sum is the determinant of ***A***.

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■ Example 3: 3 × 3 Matrix

Find the determinant of the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Expanding by the first row, the determinant can be evaluated as follows:

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■ Example 3 (cont'd): 3 × 3 Matrix

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{Cofactor of } a_{11}} + a_{12}\underbrace{(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\text{Cofactor of } a_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{Cofactor of } a_{13}}$$

Determinants

■ Example 3 (cont'd): 3 × 3 Matrix

Therefore,

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[a_{22}a_{33} - a_{23}a_{32}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Determinants

■ Example 4: 4 by 4 Matrix

Find det(***A***) if

$$A = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -1 & 4 & 1 & 0 \\ 3 & 0 & 4 & 1 \\ -2 & 1 & 1 & 3 \end{bmatrix}$$

First, check to see which row or column contains the most zeros and expand by it.
Thus expanding by the second column gives

Determinants

■ Example 4 (cont'd): 4 by 4 Matrix

$$|A| = 0(\text{cofactor of } 0) + 4(\text{cofactor of } 4) + 0(\text{cofactor of } 0) + 1(\text{cofactor of } 1)$$

$$= 0 + 4(-1)^{2+2}\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} + 0 + 1(-1)^{4+2}\begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -1 & 4 & 1 & 0 \\ 3 & 0 & 4 & 1 \\ -2 & 1 & 1 & 3 \end{bmatrix}$$

Determinants

■ Example 4 (cont'd): 4 by 4 Matrix

$$\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix}$$

$$= [(4)(3) - (1)(1)] - 5[(3)(3) - (1)(-2)] + 2[(3)(1) - (4)(-2)]$$

$$= [12 - 1] - 5[9 + 2] + 2[3 + 8]$$

$$= 11 - 5(11) + 2(11)$$

$$= -22$$

Determinants

■ <u>Example 4 (cont'd):</u> 4 by 4 Matrix

$$\begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} = 2(-1)^{1+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} + 1(-1)^{3+3}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix}$$

$$= 2[(-1)(4)-(1)(3)] + 0 + [(1)(1)-(5)(-1)]$$

$$= 2[-4-3] + [1+5]$$

$$= -14 + 6$$

$$= -8$$

Determinants

■ <u>Example 4 (cont'd):</u> 4 by 4 Matrix

Therefore,

$$|A| = 0(\text{cofactor of } 0) + 4(\text{cofactor of } 4) + 0(\text{cofactor of } 0) + 1(\text{cofactor of } 1)$$

$$= 0 + 4(-1)^{2+2}\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} + 0 + 1(-1)^{4+2}\begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1 & 5 & 2 \\ 3 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} = 4(-22) + (-8) = -88 - 8 = \underline{-96}$$

Determinants

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■ Properties of Determinants

1. If the elements of any two rows (columns) are equal, the determinant equals zero.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 14 & 2 \\ 3 & 5 & 3 \end{bmatrix}$$

$$\det(A) = |A| = 0$$

Determinants

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■ Properties of Determinants

2. If the values in any row (column) are proportional to the corresponding values in another row (column), the determinant equals zero.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 14 & 4 \\ 3 & 5 & 6 \end{bmatrix}$$

det[A] = 0 because column 3 = 2 $\times$ column 1
or the first column is proportional to the third column

Determinants

- Properties of Determinants

  3. If all the elements in any row (column) equal zero, the determinant equal zero.

$$A = \begin{bmatrix} 11 & 1 & 0 & 5 \\ 6 & 5 & 0 & 6 \\ 7 & 3 & 0 & 11 \\ 9 & 2 & 0 & 5 \end{bmatrix}$$

$$\det(A) = 0$$

Determinants

- Properties of Determinants

  4. If a matrix ***B*** is obtained from a matrix ***A*** by multiplying every element in one row (one column) of ***A*** by a constant *c*, then |***B***| = *c*|***A***|.

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(2) \\ 4 & 5 \end{bmatrix}$$

$$\det(B) = 2|A| = 2[3(5) - 2(4)] = 14$$

Determinants

- Properties of Determinants
  5. The value of the determinant is not changed by adding any row (column) multiplied by a constant *c* to another row (column).

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}, \quad |A| = 3(5) - 2(4) = 7$$

Multiplying the second row by (-1) and adding it to the first Row produces the following matrix ***B***

$$B = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix}, \quad |B| = -1(5) - (-3)(4) = 7 = |A|$$

Determinants

- Properties of Determinants
  6. If any two rows (columns) are interchanged, the sign of the determinant will be changed.

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 3(5) - 2(4) = 7 \text{ and } \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = 4(2) - 5(3) = -7$$

Determinants

- Properties of Determinants

7. For an $n \times n$ ***A*** and any constant *c*, the det(*c**A***) = c^n det(***A***).

$$A = 3^2\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}, \quad |A| = 3^2[3(5) - 2(4)] = 63$$

or

$$A = \begin{bmatrix} 3(3) & 3(2) \\ 3(4) & 3(5) \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 12 & 15 \end{bmatrix}, \quad |A| = [9(15) - 6(12)] = 63$$

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- Properties of Determinants

8. The determinant of square matrix equals that of its transpose, that is, |***A***| = |$\boldsymbol{A}^T$|

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}, \quad |A| = 3(5) - 2(4) = 7$$

$$A^T = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}, \quad |B| = (3)5 - 4(2) = 7$$

Determinants

■ Properties of Determinants

9. If a square matrix ***A*** is placed in the diagonal form using property 5, then the product of the elements on the diagonal equals the determinant of ***A***

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}, \quad |A| = 3(5) - 2(4) = 7$$

Multiplying the first row by –4/3 and adding it to the second Row produces a matrix with a zero element in the second Row and first column as follows:

Determinants

$$\begin{bmatrix} 3 & 2 \\ 0 & \frac{7}{3} \end{bmatrix}$$

– Then, multiplying the second row by –6/7 and adding it to the first row results in the following diagonal matrix:

$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{7}{3} \end{bmatrix}$$

– Therefore, the determinant of ***A*** is

$$|A| = \begin{vmatrix} 3 & 0 \\ 0 & \frac{7}{3} \end{vmatrix} = 3\left(\frac{7}{3}\right) = 7$$

Determinants

- Properties of Determinants

  10. If a matrix ***A*** has a zero determinant, then ***A*** is said to be a singular matrix, that is, the inverse of ***A*** does not exist.

Rank of a Matrix

- Definition
  – The rank of a matrix ***A***, designated $r(\boldsymbol{A})$, is the order of the largest nonzero minor of ***A***.

Rank of a Matrix

■ Example 1: Rank of a Matrix

Find the rank of

$$A = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}$$

The largest minor that can be formed from ***A*** is of order 3. There is only one such minor, namely det(***A***), and it is zero. Thus, the rank of ***A*** will be 2 or less. Checking all the 9 minors of order 2, we find that each of them is also equal to zero.

Rank of a Matrix

■ Example (cont'd): Rank of a Matrix

$$\text{i.e.,} \begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} = 1(3) - 3(-1) = 0$$

$$\text{i.e.,} \begin{vmatrix} -1 & 4 \\ 2 & -8 \end{vmatrix} = -1(-8) - 4(2) = 0$$

Hence, the rank of ***A*** will be 1 or zero.
Checking minors of order 1, we find that one which is not zero (in fact all are nonzero); therefore,

$$r(A) = 1$$

Rank of a Matrix

- Example 2: Rank of a Matrix

  Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 5 & -4 & 3 \\ 3 & 2 & 11 & -4 & 5 \\ 1 & -4 & -1 & 8 & -3 \end{bmatrix}$$

All 10 minors of order 3 equal zero, so the rank of $\boldsymbol{A}$ will be 2 or less. Checking all minors of order 2, we find one of them, namely

$$\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} = 3(-4) - 2(1) = -14$$

differs from zero, so $r(\boldsymbol{A}) = 2$

Inverse of a Matrix by Cofactor and Adjoint Matrices

- Definition

  The *cofactor matrix* associated with an $n \times n$ matrix $\boldsymbol{A}$ is an $n \times n$ matrix $\boldsymbol{A}^c$ obtained from $\boldsymbol{A}$ by replacing each element of $\boldsymbol{A}$ by its cofactor.

Inverse of a Matrix by Cofactor and Adjoint Matrices

■ Example: Cofactor Matrix

What is the cofactor matrix of ***A***, if

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix}$$

Inverse of a Matrix by Cofactor and Adjoint Matrices

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■ Example (cont'd): Cofactor Matrix

$$A^c = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -2 & 4 \\ 1 & 6 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 1 & 6 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 16 & -11 \\ 0 & 16 & -8 \\ -6 & -16 & 17 \end{bmatrix}$$

Inverse of a Matrix by Cofactor and Adjoint Matrices

■ Definition

The *adjoint* of an $n \times n$ matrix $\boldsymbol{A}$ is the transpose of the cofactor matrix of $\boldsymbol{A}$.

If the adjoint of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^a$, then

$$A^a = \left(A^c\right)^T$$

Inverse of a Matrix by Cofactor and Adjoint Matrices

■ Example: Adjoint of a Matrix

Find $\boldsymbol{A}^a$ for matrix $\boldsymbol{A}$ given the previous example.

$$\text{From previous example } A^c = \begin{bmatrix} 18 & 16 & -11 \\ 0 & 16 & -8 \\ -6 & -16 & 17 \end{bmatrix}$$

$$\text{Therefore, } A^a = \left(A^c\right)^T = \begin{bmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{bmatrix}$$

Inverse of a Matrix by Cofactor and Adjoint Matrices

■ Theorem

If $|\boldsymbol{A}| \neq 0$, then the inverse of ***A*** may be obtained by dividing the adjoint of ***A*** by the determinant of ***A***, that is

$$A^{-1} = \frac{A^a}{|A|}$$

Inverse of a Matrix by Cofactor and Adjoint Matrices

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■ Example: Inverse of a Matrix

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\det(A) = 2(7) - 3(5) = -1$$

$$A^c = \begin{bmatrix} (-1)^{1+1}(7) & (-1)^{1+2}(5) \\ (-1)^{2+1}(3) & (-1)^{2+2}(2) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^a = \left(A^c\right)^T = \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{A^a}{|A|} = \frac{\begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}}{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$