CHAPTER 2c.
MATRICES

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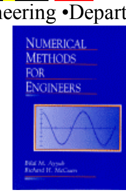

by

Dr. Ibrahim A. Assakkaf

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering
University of Maryland, College Park

Matrix Operations

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- Matrix Inversion
  - The inverse of an $n \times n$ matrix $\boldsymbol{A}$ is an $n \times n$ matrix $\boldsymbol{B}$ having the property that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$$

  - $\boldsymbol{B}$ is called the inverse of $\boldsymbol{A}$ and is usually denoted by $\boldsymbol{A}^{-1}$.
  - Hence,

$$\boldsymbol{A}^{-1}\,\boldsymbol{A} = \boldsymbol{I}$$

Matrix Operations

- Matrix Inversion
  - Properties:
    - If a square matrix ***A*** has an inverse, it is said to be *invertible* or *nonsingular*.
    - If it dose not possess an inverse, it is *singular*.
    - In particular, the identity or unit matrix ***I*** is invertible and is its own inverse since

$$II = I$$

Matrix Operations

- Matrix Inversion
  - The inverse can be determined by forming n^2 simultaneous equations and solving for n^2 unknowns.
  - Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Matrix Operations

- Matrix Inversion
  - Therefore,

$$A^{-1}A = I$$

$$A^{-1}A = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} c_{11}a_{11} + c_{12}a_{21} & c_{11}a_{12} + c_{12}a_{22} \\ c_{21}a_{11} + c_{22}a_{21} & c_{21}a_{12} + c_{22}a_{22} \end{bmatrix}$$

or

$$\begin{bmatrix} c_{11}a_{11} + c_{12}a_{21} & c_{11}a_{12} + c_{12}a_{22} \\ c_{21}a_{11} + c_{22}a_{21} & c_{21}a_{12} + c_{22}a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Operations

- Matrix Inversion
  - Hence, the following for simultaneous equations can be solved for c_{ij} given the values a_{ij} of the original matrix ***A***:

$$\begin{aligned} c_{11}a_{11} + c_{12}a_{21} &= 1 \\ c_{11}a_{12} + c_{12}a_{22} &= 0 \\ c_{21}a_{11} + c_{22}a_{21} &= 0 \\ c_{21}a_{12} + c_{22}a_{22} &= 1 \end{aligned}$$

Matrix Operations

■ Example 1: Matrix Inversion

Find $\boldsymbol{A}^{-1}$ if $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

$$A^{-1}A = I$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2c_{11} + 5c_{12} & 3c_{11} + 7c_{12} \\ 2c_{21} + 5c_{22} & 3c_{21} + 7c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Operations

■ Example 1 (cont'd): Matrix Inversion

$$2c_{11} + 5c_{12} = 1$$

$$3c_{11} + 7c_{12} = 0$$

$$2c_{21} + 5c_{22} = 0$$

$$3c_{21} + 7c_{22} = 1$$

From which,

$c_{11} = -7, c_{12} = 3, c_{21} = 5,$ and $c_{22} = -2$

Therfore, $A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$

Matrix Operations

■ Example 2: Matrix Inversion

Find $\boldsymbol{A}^{-1}$ if $A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix}$

$$A^{-1}A = I$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3c_{11} - 2c_{12} + c_{13} & c_{11} + 5c_{12} + 3c_{13} & 2c_{11} + 4c_{12} + 6c_{13} \\ 3c_{21} - 2c_{22} + c_{13} & c_{21} + 5c_{22} + 3c_{23} & 2c_{21} + 4c_{22} + 6c_{23} \\ 3c_{31} - 2c_{32} + c_{33} & c_{31} + 5c_{32} + 3c_{33} & 2c_{31} + 4c_{32} + 6c_{33} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Operations

■ Example 2 (cont'd): Matrix Inversion

$$\begin{aligned} 3c_{11} - 2c_{12} + c_{13} &= 1 \\ c_{11} + 5c_{12} + 3c_{13} &= 0 \\ 2c_{11} + 4c_{12} + 6c_{13} &= 0 \end{aligned}$$

$$\begin{aligned} 3c_{21} - 2c_{22} + c_{13} &= 0 \\ c_{21} + 5c_{22} + 3c_{23} &= 1 \\ 2c_{21} + 4c_{22} + 6c_{23} &= 0 \end{aligned}$$

$$\begin{aligned} 3c_{31} - 2c_{32} + c_{33} &= 0 \\ c_{31} + 5c_{32} + 3c_{33} &= 0 \\ 2c_{31} + 4c_{32} + 6c_{33} &= 1 \end{aligned}$$

3 Sets of Simultaneous Equations

Matrix Operations

■ Example 2 (cont'd): Matrix Inversion

$$\begin{aligned} 3c_{11} - 2c_{12} + c_{13} &= 1 \\ c_{11} + 5c_{12} + 3c_{13} &= 0 \\ 2c_{11} + 4c_{12} + 6c_{13} &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} c_{11} &= 0.3750 \\ c_{12} &= 0 \\ c_{13} &= -0.1250 \end{aligned}$$

$$\begin{aligned} 3c_{21} - 2c_{22} + c_{13} &= 0 \\ c_{21} + 5c_{22} + 3c_{23} &= 1 \\ 2c_{21} + 4c_{22} + 6c_{23} &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} c_{21} &= 0.3333 \\ c_{22} &= 0.3333 \\ c_{23} &= -0.3333 \end{aligned}$$

Matrix Operations

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■ Example 2 (cont'd): Matrix Inversion

$$\begin{aligned} 3c_{31} - 2c_{32} + c_{33} &= 0 \\ c_{31} + 5c_{32} + 3c_{33} &= 0 \\ 2c_{31} + 4c_{32} + 6c_{33} &= 1 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} c_{31} &= -0.2292 \\ c_{32} &= -0.1667 \\ c_{33} &= 0.3542 \end{aligned}$$

Therefore,

$$A^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0.3750 & 0 & -0.1250 \\ 0.3333 & 0.3333 & -0.3333 \\ -0.2292 & -0.1667 & 0.3542 \end{bmatrix}$$

Matrix Operations

- Matrix Singularity
  - If the inverse of a matrix exists, then the matrix is ***nonsingular***.
  - If the inverse does not exist, then the matrix is ***singular***.

Matrix Operations

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- Matrix Singularity
  - Matrix Singularity and System of Equations
    - One implication of matrix singularity in solving a system of simultaneous equations is that a *unique solution* for the equation does not exist.
    - If the matrix is singular, the system of equation will not have a solution.

Matrix Operations

- Matrix Singularity
  - Matrix Singularity and System of Equations

$$2X_1 + 3X_2 = a$$
$$4X_1 + 6X_2 = b$$

If $2a = b$, then there are an infinite number of solutions. For example, three possibilities are:

1) $X_1 = 2$, $X_2 = 1$, $a = 7$, and $b = 14$
2) $X_1 = -1$, $X_2 = 4$, $a = 10$, and $b = 20$
3) $X_1 = 0$, $X_2 = -2$, $a = -6$, and $b = -12$

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- Matrix Singularity
  - Matrix Singularity and System of Equations

$$2X_1 + 3X_2 = a \qquad A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$
$$4X_1 + 6X_2 = b$$

If $a \neq b$, then there is no feasible solution.

For example, if $a = 2$ and $b = 3$, there are no values of X_1 and X_2 that can satisfy the equality of the two equations.

Matrix Operations

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- Trace of a Matrix
  - The trace of square matrix is the sum of the diagonal elements as defined by

$$\boxed{tr(A) = \sum_{i=1}^{n} a_{ii}}$$

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- Trace of a Matrix
  - Example:

    Find the trace of the following matrices:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 11 & 3 \\ -5 & -5 \end{bmatrix}$$

$$tr(A) = 3 + 5 + 6 = 14$$
$$tr(B) = 11 + (-5) = 6$$

Matrix Operations

- Matrix Augmentation
  - Matrix augmentation is the addition of a column or columns to the initial matrix

$$\left[\begin{array}{ccccc|ccccc} a_{11} & a_{12} & a_{13} & \dots & a_{1c} & c_{11} & c_{12} & c_{13} & \dots & c_{1c} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2c} & c_{21} & c_{22} & c_{23} & \dots & c_{2c} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3c} & c_{31} & c_{32} & c_{33} & \dots & c_{3c} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & a_{r3} & \dots & a_{rc} & c_{r1} & c_{r2} & c_{r3} & \dots & c_{rc} \end{array}\right]$$

Matrix Operations

- Examples: Matrix Augmentation

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 4 & 1 \\ 2 & 3 & 10 \end{bmatrix}$$

$$A_a = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \qquad B_a = \left[\begin{array}{ccc|c} 0 & 3 & 1 & 2 \\ 1 & 4 & 1 & 3 \\ 2 & 3 & 10 & 6 \end{array}\right]$$

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■ Submatrices and Partitioning

– Given any matrix ***A***, a *submatrix* of ***A*** is a matrix obtained from ***A*** by removing any number of rows or columns.

– Thus if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

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■ Submatrices and Partitioning

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

– Then ***B*** and ***C*** are both submatrices of ***A***

– ***B*** is obtained by removing from ***A*** the first and second rows together with the first and third columns

– ***C*** is obtained by by removing from ***A*** the second, third, and fourth rows together with the first column.

Matrix Operations

■ Submatrices and Partitioning

– A matrix can be partitioned by separating it into smaller matrices.

– For example, matrix ***A*** can be partitioned into four other matrices as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

Matrix Operations

■ Example: Matrix Partition

$$A = \begin{bmatrix} 1 & 0.2 & 0.9 \\ 0.2 & 1 & 0.8 \\ 0.9 & 0.8 & 1 \end{bmatrix} \Longrightarrow A = \left[\begin{array}{cc|c} 1 & 0.2 & 0.9 \\ 0.2 & 1 & 0.8 \\ \hline 0.9 & 0.8 & 1 \end{array}\right]$$

$$A_{11} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 0.9 \\ 0.8 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0.9 & 0.8 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 1 \end{bmatrix}$$

Vectors

- Definitions
  - A vector is a $1 \times n$ or $n \times 1$ matrix
  - A $1 \times n$ matrix is called a *row vector*
  - An $n \times 1$ matrix is called a *column vector*
  - The elements are called the *components* of the vector.
  - The number of components, in this case, is its *dimension*.

Vectors

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- Examples: Vectors

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad V_2 = \begin{bmatrix} t & 2t & -t & 0 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} -0.5 & 4 \end{bmatrix} \qquad Z_2 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}$$

Vectors

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- Vectors in Space
  - The name vector indicates that a one-dimensional matrix with *n* elements can be represented as a vector in *n*-dimensional space, with one end of the vector at a point and the the other end at another point.

$$V_{P_2-P_1} = \left[\left(P_{2X_1} - P_{1X_1}\right) \quad \left(P_{2X_2} - P_{1X_2}\right) \quad \left(P_{2X_3} - X_{1X_3}\right) \quad \ldots \quad \left(P_{2X_n} - P_{1X_n}\right) \right]$$

Vectors

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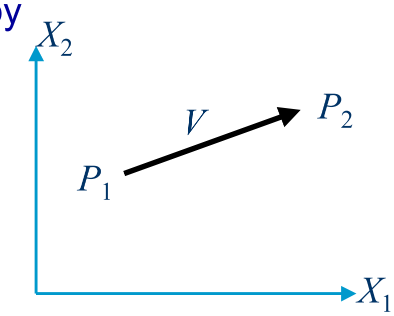
- Vector in Two-dimensional Space
  - In general the vector in two-dimensional space is given by

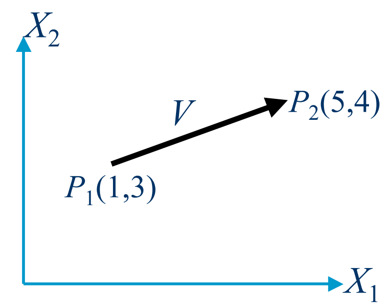
$$V_{P_2-P_1} = \left[\left(P_{2X_1} - P_{1X_1}\right) \quad \left(P_{2X_2} - P_{1X_2}\right) \right]$$



Vectors

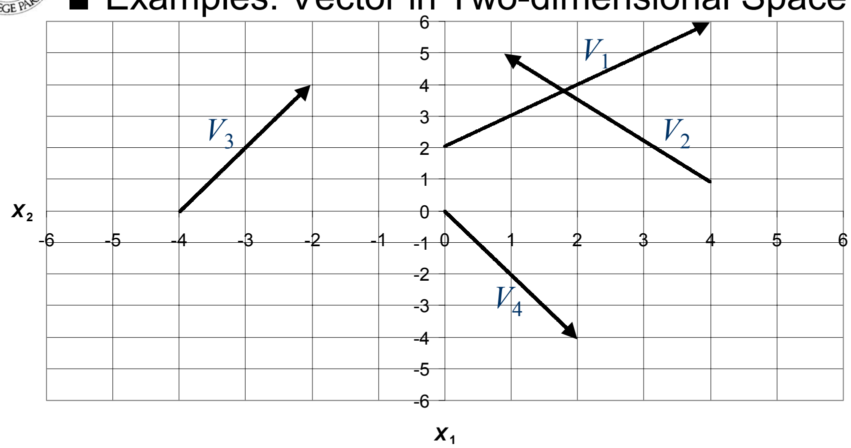
- Example: Vector in Two-dimensional Space

$$V_{P_2-P_1} = \left[\left(P_{2X_1} - P_{1X_1}\right) \quad \left(P_{2X_2} - P_{1X_2}\right)\right]$$
$$= \left[(5-1) \quad (4-3)\right]$$
$$= \left[4 \quad 1\right]$$



Vectors

- Examples: Vector in Two-dimensional Space



Vectors

- Examples: Vector in Two-dimensional Space

$$V_1 = [(4-0) \quad (6-2)] = [4 \quad 4]$$
$$V_2 = [(1-4) \quad (5-1)] = [-3 \quad 4]$$
$$V_3 = [(-2-(-4)) \quad (4-0)] = [2 \quad 4]$$
$$V_4 = [(2-0) \quad (-4-0)] = [4 \quad -4]$$

Vectors

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- Vector Operations
  – The matrix operations of addition, subtraction, and multiplication can be applied to vectors.
  – Two row (or column) vectors can be added or subtracted.
  – A row vector with *n* elements can be postmultiplied by a column vector with *n* elements to equal a scalar value

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■ Example: Vector Operations

Find $\boldsymbol{V}_1 + 3\boldsymbol{V}_2$, $\boldsymbol{V}_2 - 2\boldsymbol{V}_1$, and $\boldsymbol{V}_1^T\boldsymbol{V}_2$ if

$$V_1 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

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■ Example (cont'd): Vector Operations

$$V_1 + 3V_2 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \\ 12 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 18 \\ 3 \end{bmatrix}$$

$$V_2 - 2V_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ -8 \\ 1 \end{bmatrix}$$

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■ Example (cont'd): Vector Operations

$$V_1^T = [3 \quad 1 \quad 6 \quad 0]$$

$$V_1^T V_2 = [3 \quad 1 \quad 6 \quad 0]\begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} = [3(2)+1(-1)+6(4)+(0)(1)] = [29]$$

$(1 \times 4) \quad (4 \times 1)$

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■ Orthogonal and Normalized Vectors

– Two vectors are said to be *orthogonal* if their product equals zero.

– For the vector product ***A•B***, ***A*** is a row vector and ***B*** is a column vector, the resulting vector product is a scalar value.

– If two vectors that are orthogonal are plotted in the *n*-dimensional space, the vectors will be perpendicular to each other.

Vectors

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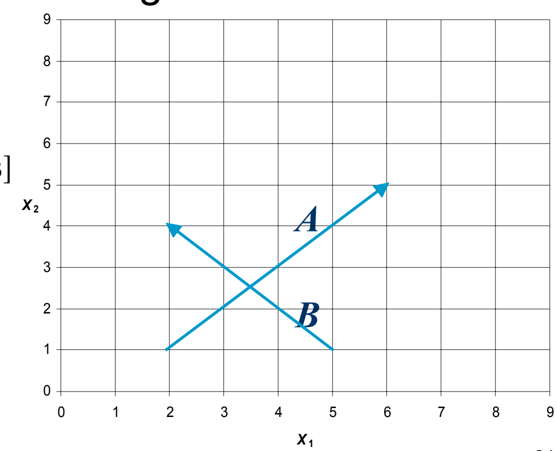
■ Example: Orthogonal Vectors

$$A = [(6-2) \quad (5-1)] = [4 \quad 4]$$
$$B = [(2-5) \quad (4-1)] = [-3 \quad 3]$$



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■ Example: Orthogonal Vectors

$$A \bullet B = [4 \quad 4]\begin{bmatrix} -3 \\ 3 \end{bmatrix} = [-12 + 12] = [0]$$

Since the vector product is zero, the vectors are perpendicular (orthogonal) to each other.

Vectors

■ Length (magnitude) of a Vector

The *length* or *magnitude* of a vector ***V*** equals the square root of the sum of the squares of its elements, that is

$$\text{Length of } V = \|V\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Vectors

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■ Normalized Vector

- A vector is said to be *normalized* if each elements of the vector is divided by its length.
- A normalized vector has a length that is equal to one.
- A *unit* vector is also a normalized vector.

Vectors

■ Orthonormal Vectors

Two vectors that are both normalized and orthogonal to each other are said to *orthonormal* vectors

Vectors

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■ Example

For the following vectors $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, perform the following:

1. Find their lengths (magnitudes)
2. Normalize them,
3. State whether they are orthonormal

$$V_1 = \begin{bmatrix} 2 & -3 & 5 \end{bmatrix} \qquad V_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Vectors

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■ Example (cont'd)

$$\text{Length of } V_1 = \sqrt{(2)^2 + (-3)^2 + (5)^2} = \sqrt{38} = 6.164$$

$$\text{Length of } V_2 = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3} = 1.732$$

The normalized vectors are:

$$V_{n1} = \begin{bmatrix} \frac{2}{\sqrt{38}} & \frac{-3}{\sqrt{38}} & \frac{5}{\sqrt{38}} \end{bmatrix}$$

$$V_{n2} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Since the $\boldsymbol{V}_1\boldsymbol{V}_2 = 0$, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are orthogonal.
$\boldsymbol{V}_{n1}$ and $\boldsymbol{V}_{n2}$ are orthonormal.