

CHAPTER 2a. MATRICES



• A. J. Clark School of Engineering • Department of Civil and Environmental Engineering



by

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Introduction



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■ Definition of a Matrix

A matrix is a rectangular array of elements arranged in a horizontal rows and vertical columns.



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Examples:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \text{republican} & 37 & \text{good} & \text{not qualified} \\ \text{democrat} & 48 & \text{fair} & \text{qualified} \end{bmatrix}$$

$$\begin{bmatrix} 4\theta_1 & \theta_2 & -\frac{3}{L}d & \frac{wL^3}{24E_1} \\ \theta_1 & 4\theta_2 & -\frac{3}{L}d & \frac{-wL^3}{24E_1} \\ \theta_1 & \theta_2 & -\frac{4}{L}d & \frac{PL^2}{6EI} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x^2 & x \\ e^x & \frac{d}{dx} \ln x \\ 5 & x+2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} \\ \pi \\ 19.5 \end{bmatrix}$$



Introduction

General Form of a Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1c} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2c} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3c} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & a_{r3} & \dots & a_{rc} \end{bmatrix} \quad (1)$$



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■ General Form of a Matrix

- Each column has the same number of values r
 r = number of rows
- Each row has the same number of values c
 c = number of columns
- It is not necessary for the number of rows to equal the number of columns



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■ General From of a Matrix

- Notations and Convention
 - A capital letter can be used to denote the name of an array or a matrix.
 - The values contained in the matrix are called elements and are denoted as a_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix} \quad M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{33} \end{bmatrix}$$



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■ General Form of a Matrix

– Notations and Convention

- A matrix can also be abbreviated using the following notations:

$$\left[a_{ij} \right]_{r \times c} \quad \text{or simply} \quad \left[a_{ij} \right]$$

- In this notation, a_{ij} represents the general element of the matrix and appears in the i^{th} row and j^{th} column.
- The subscript i , which represents the row, can have any value 1 through r .



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■ General Form of a Matrix

– Notations and Convention

- The subscript j , which represents the column, can have values from 1 to c .
- Thus, if $i = 2$ and $j = 3$, a_{ij} becomes a_{23} and designates the element in the second row and third column.
- if $i = 1$ and $j = 5$, a_{ij} becomes a_{15} and designates the element in the first row and fifth column.



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■ Order or Size of a Matrix

– The order (size) of a matrix is given by

$$\text{Order} = r \times c$$

By convention, $r \times c$ is read r by c , and the row index is always given first.



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■ Order or Size of a Matrix

– Examples:

$$\begin{bmatrix} x^2 & x \\ e^{-x} & \frac{d}{dx} \ln x \\ 5 & x+2 \end{bmatrix}$$

Size: 3×3

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

Size: 2×3

$$\begin{bmatrix} \sqrt{2} \\ \pi \\ 19.5 \end{bmatrix}$$

Size: 3×1

$$[\sqrt{2} \quad \pi \quad 19.5]$$

Size: 1×3

$$\begin{bmatrix} 4\theta_1 & \theta_2 & -\frac{3}{L}d & \frac{wL^3}{24E} \\ \theta_1 & 4\theta_2 & -\frac{3}{L}d & -\frac{wL^3}{24E} \\ \theta_1 & \theta_2 & -\frac{4}{L}d & \frac{PL^2}{6EI} \end{bmatrix}$$

Size: 3×4



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General Form of a Matrix – Example Notations

$$A = [a_{ij}] = \begin{bmatrix} 4\theta_1 & \theta_2 & -\frac{3}{L}d & \frac{wL^3}{24E_1} \\ \theta_1 & 4\theta_2 & -\frac{3}{L}d & \frac{-wL^3}{24E_1} \\ \theta_1 & \theta_2 & -\frac{4}{L}d & \frac{PL^2}{6EI} \end{bmatrix}$$

Diagram illustrating the general form of a matrix $A = [a_{ij}]$ with specific elements circled and labeled:

- a_{22} points to the element $4\theta_2$.
- a_{33} points to the element $-\frac{4}{L}d$.
- a_{14} points to the element $\frac{wL^3}{24E_1}$.



Formation of a Matrix

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What Produces a Matrix?

- Matrices are useful for representing sets of data.
- The form of a matrix depends on the style or use.
- The form may also be dictated by computational requirements.
- The elements & variables can be properties that vary with time, space, or on the basis of any characteristic.



Formation of a Matrix

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■ What Produces a Matrix?

– The following matrix is a data matrix of concentration of three water-quality indicators:

	H	A	pH
H	1.00	-0.23	-0.27
A	-0.23	1.00	0.64
pH	-0.27	0.64	1.00

H = hardness
A = alkalinity



Formation of a Matrix

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■ Style and Use

5 × 3 matrix

3 × 5 matrix

	H	A	pH
1	140	35	7.7
2	195	12	7.1
well 3	283	53	6.4
4	132	188	8.3
5	60	55	6.5

	1	2	3	4	5
H	140	195	283	132	60
A	35	12	53	188	55
pH	7.7	7.1	6.4	8.3	6.5



Formation of a Matrix

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- What Produces a Matrix?
 - Results of matrix computations

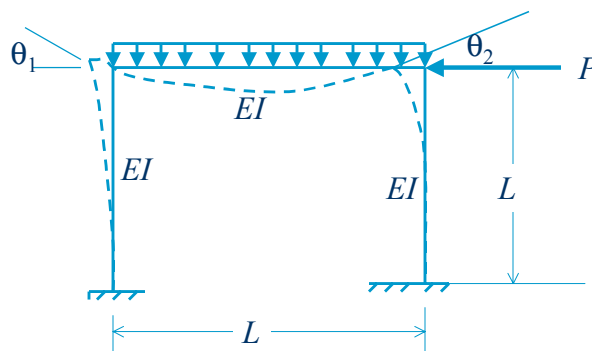
$$\begin{bmatrix} 4 & 6 & 1 \\ 3 & 8 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 7 \\ 2 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 35 & 55 \\ 43 & 81 \end{bmatrix}$$



Formation of a Matrix

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- What Produces a Matrix?
 - Result from physical structure of a problem

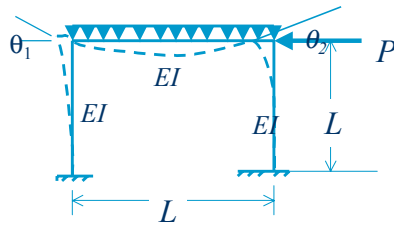




Formation of a Matrix

■ What Produces a Matrix?

– Result from physical structure of a problem



$$\begin{aligned}
 4\theta_1 \quad \theta_2 \quad -\frac{3}{L}d &= \frac{wL^3}{24E} \\
 \theta_1 \quad 4\theta_2 \quad -\frac{3}{L}d &= \frac{-wL^3}{24E} \\
 \theta_1 \quad \theta_2 \quad -\frac{4}{L}d &= \frac{PL^2}{6EI}
 \end{aligned}$$



Formation of a Matrix

■ What Produces a Matrix?

– Result from physical structure of a problem

$$\begin{aligned}
 4\theta_1 \quad \theta_2 \quad -\frac{3}{L}d &= \frac{wL^3}{24E} \\
 \theta_1 \quad 4\theta_2 \quad -\frac{3}{L}d &= \frac{-wL^3}{24E} \\
 \theta_1 \quad \theta_2 \quad -\frac{4}{L}d &= \frac{PL^2}{6EI}
 \end{aligned}
 \implies
 \begin{bmatrix}
 4\theta_1 & \theta_2 & -\frac{3}{L}d & \frac{wL^3}{24E} \\
 \theta_1 & 4\theta_2 & -\frac{3}{L}d & \frac{-wL^3}{24E} \\
 \theta_1 & \theta_2 & -\frac{4}{L}d & \frac{PL^2}{6EI}
 \end{bmatrix}$$



Types of Matrices

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The following list provides different types of matrices:

1. Square Matrix
2. Upper Triangular Matrix
3. Strictly Upper Triangular Matrix
4. Lower Triangular Matrix
5. Strictly Lower Triangular Matrix
6. Diagonal Matrix



Types of Matrices

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7. Banded Matrix
8. Unit (identity) Matrix
9. Null Matrix
10. Symmetric Matrix
11. Skew-symmetric Matrix
12. Transposed Matrix



Types of Matrices

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■ Square Matrix

– A *square matrix* is a special case in which the number of rows equals the number of columns

– Examples

$$\begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \quad \begin{bmatrix} G & H \\ W & B \end{bmatrix} \quad [4.76]$$



Types of Matrices

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■ Square Matrix

– The set of elements a_{ij} for which $i = j$ is called the “principal diagonal”

$$\begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$



Types of Matrices

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■ Upper Triangular Matrix

- An *upper triangular matrix* has values on the principal diagonal and above, but contains zero values for the matrix elements below the principal diagonal.

- Examples:

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 4 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

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Types of Matrices

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■ Strictly Upper Triangular Matrix

- A *strictly upper triangular matrix* is an upper triangular matrix with the elements of the principal diagonal have zero values.

- Examples:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Types of Matrices

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■ Lower Triangular Matrix

- An *lower triangular matrix* has values on the principal diagonal and below, but contains zero values for the matrix elements above the principal diagonal.

- Examples:

$$\begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 7 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 1 & 4 & 1 \end{bmatrix}$$



Types of Matrices

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■ Strictly Lower Triangular Matrix

- A *strictly lower triangular matrix* is an lower triangular matrix with the elements of the principal diagonal have zero values.

- Examples:

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 7 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 \end{bmatrix}$$



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■ Diagonal Matrix

– A *diagonal matrix* is a square matrix with all the elements equal to zero except for the elements on the principal diagonal.

– Examples:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1.2 & 0 \\ 0 & 0 & 7.1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$



Types of Matrices

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■ Banded Matrix

– A *banded matrix* is a square matrix with all the elements equal to zero except for the principal diagonal and values in the positions adjacent to the principal diagonal.

- A *tridiagonal matrix* is the especial case of a banded matrix that has zeros except in the principal diagonal and the two adjacent diagonals



Types of Matrices

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– Examples: Banded Matrices

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 2 & 7 & 4 & 0 & 0 \\ 0 & 3 & 8 & 3 & 0 \\ 0 & 0 & 1 & 10 & 5 \\ 0 & 0 & 0 & 6 & 15 \end{bmatrix}$$

A tridiagonal banded matrix

$$\begin{bmatrix} 0.5 & 2 & 7 & 0 & 0 & 0 \\ 2 & 1 & 3 & 2 & 0 & 0 \\ 7 & 6 & 2 & 5 & 1 & 0 \\ 0 & 1 & 2 & 4 & 9 & 3 \\ 0 & 0 & 4 & 2 & 5 & 2 \\ 0 & 0 & 0 & 6 & 9 & 7 \end{bmatrix}$$

A banded matrix