Homework #10 Solution
ENCE 203 - Spring 2001
Due F, 5/11

<u>Problem1:</u>
For the following integral:

$$\int_0^3 \frac{e^x \sin x}{1+x^2}dx$$

estimate its value using:

(a) Two and three-point Gauss Quadrature.
(b) Simpson 1/3-rule with 5 points (four intervals).
(c) Compare the results of part (a) and (b) with the exact solution of 2.881637.

*** SOLUTION ***

(a) Two and Three-point Gauss Quarature:

$$\int_0^3 \frac{e^x \sin x}{1+x^2}dx \quad \Rightarrow f(x) = \frac{e^x \sin x}{1+x^2}$$

$$x = \frac{b+a}{2} + \frac{b-a}{2}x_G \Rightarrow \frac{3+0}{2} + \frac{3-0}{2}x_G \Rightarrow 1.5 + 1.5x_G$$

$$dx = \frac{b-a}{2}dx_G \Rightarrow dx = 1.5dx_G$$

$$\int_a^b f(x)dx \Rightarrow \int_{-1}^{b1} f(x_G)dx_G$$

$$\int_0^3 \frac{e^x \sin x}{1+x^2}dx = \int_{-1}^{1} \frac{e^{1.5+1.5x_G}\sin(1.5+1.5x_G)}{1+(1.5+1.5x_G)^2}(1.5dx_G)$$

For two-point:

$$I = \int_{-1}^{1} f(x_G)dx_G = C_1 f(x_1) + C_2 f(x_2)$$

From Gauss Quadrature Table:

$C_1 = 1 \qquad x_1 = -0.5774$

$C_2 = 1 \qquad x_2 = 0.5774$

$$f(x_G) = \frac{e^{1.5+1.5x_G}\sin(1.5+1.5x_G)}{1+(1.5+1.5x_G)^2}(1.5)$$

$$f(-0.5774) = \frac{e^{1.5+1.5(-0.5774)}\sin[1.5+1.5(-0.5774)]}{1+[1.5+1.5(-0.5774)]^2}(1.5) = 1.194624$$

$$f(0.5774) = \frac{e^{1.5+1.5(0.5774)}\sin[1.5+1.5(0.5774)]}{1+[1.5+1.5(0.5774)]^2}(1.5) = 1.694624$$

Therefore,

$$I_{2p} = C_1 f(x_1) + C_2 f(x_2) = 1(1.194624) + 1(1.695801) = 2.890425 \quad \leftarrow \textbf{Ans}$$

For three-point:

$$I = \int_{-1}^{1} f(x_G)dx_G = C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

From Gauss Quadrature Table:

$$C_1 = 0.5555 \qquad x_1 = -0.7746$$
$$C_2 = 0.8888 \qquad x_2 = 0$$
$$C_1 = 0.5555 \qquad x_1 = 0.7746$$

$$f(x_G) = \frac{e^{1.5+1.5x_G}\sin(1.5+1.5x_G)}{1+(1.5+1.5x_G)^2}(1.5)$$

$$f(-0.7746) = \frac{e^{1.5+1.5(-0.7746)}\sin[1.5+1.5(-0.7746)]}{1+[1.5+1.5(-0.7746)]^2}(1.5) = 0.626123$$

$$f(0) = \frac{e^{1.5+1.5(0)}\sin[1.5+1.5(0)]}{1+[1.5+1.5(0)]^2}(1.5) = 2.063291$$

$$f(0.7746) = \frac{e^{1.5+1.5(0.7746)}\sin[1.5+1.5(0.7746)]}{1+[1.5+1.5(0.7746)]^2}(1.5) = 1.226307$$

Therefore,

$$I_{2\text{p}} = C_1 f(x_1) + C_2 f(x_2) + C_2 f(x_3) =$$
$$= 0.5555(0.626123) + 0.8888(2.063291) + 0.5555(1.226307)$$
$$= 2.862878 \leftarrow \textbf{Ans}$$

(b) Simpson's 1/3-rule with 5 points:

i	1	2	3	4	5
x	0	0.75	1.5	2.25	3.0
f(*x*)	0	0.923539	1.375527	1.217675	0.283447

$$I_{1/3-\text{S}_{5\text{-point}}} = \frac{0.75}{3}\left[\{0+4(1.923539)+1.375527\} + \{1.375526+4(1.217675)+0.283447\}\right]$$

$$= 2.899839 \leftarrow \textbf{Ans}$$

(c) Comparison with the Exact Solution:

$$\text{Error } I_{\text{G2P}} = \left|\frac{2.881637 - 2.890425}{2.881637}\right| \times 100 = 0.3\%$$

$$\text{Error } I_{\text{G3P}} = \left|\frac{2.881637 - 2.862878}{2.881637}\right| \times 100 = 0.65\%$$

$$\text{Error } I_{\text{S5P}} = \left|\frac{2.881637 - 2.899839}{2.881637}\right| \times 100 = 0.63\%$$

Problem 2:

Textbook: 8-6

$$\frac{dy}{dx} = e^x + xe^x + 1 \qquad \text{such that } y = 0 \text{ at } x = 0$$

$x_0 = 0$

$$g(x_0) = y(0) = g(0) = 0$$

$$g'(x) = \frac{dy}{dx} = e^x + xe^x + 1 \qquad \Rightarrow \qquad g'(0) = 2$$

$$g''(x) = \frac{d^2y}{dx^2} = 2e^x + xe^x \qquad \Rightarrow \qquad g''(0) = 2$$

$$g'''(x) = \frac{d^3y}{dx^3} = 3e^x + xe^x \qquad \Rightarrow \qquad g'''(0) = 3$$

$$g''''(x) = \frac{d^4y}{dx^4} = 4e^x + xe^x \qquad \Rightarrow \qquad g''''(0) = 4$$

$$g'''''(x) = \frac{d^5y}{dx^5} = 5e^x + xe^x \qquad \Rightarrow \qquad g'''''(0) = 5$$

Using Eq. 8-9 of the textbook, the following six-terms Tayler series expansion can be obtained:

$$g(x) = 0 + (x-0)(2) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(3) + \frac{(x-0)^4}{4!}(4) + \frac{(x-0)^5}{5!}(5)$$

For $x = 0.2$

$$g(0.2) = 0 + (0.2)(2) + \frac{(0.2)^2}{2}(2) + \frac{(0.2)^3}{6}(3) + \frac{(0.2)^4}{24}(4) + \frac{(0.2)^5}{120}(5) = 0.444280$$

The following table summarizes the estimated values of the function $y(x) = g(x)$ for the range of $x = 0$ to 1 in an increment of 0.2. The table also compares the computed values with the true values.

x	Estimated $y(x)$	True $y(x)$	error(%)
0	0	0	0
0.2	0.444280	0.444281	0.00012
0.4	0.996693	0.996730	0.00367
0.6	1.692840	1.693271	0.02547
0.8	2.577920	2.580433	0.09738
1	3.708333	3.718282	0.26756

Problem 3:

Textbook: 8-8

$\Delta x = h$ 0.2

x	y'	y-Euler	y-exact	error(%)
0	-1	2	2	0.00
0.2	-0.8	1.8	1.818731	-1.03
0.4	-0.64	1.64	1.67032	-1.82
0.6	-0.512	1.512	1.548812	-2.38
0.8	-0.4096	1.4096	1.449329	-2.74
1	-0.32768	1.32768	1.367879	-2.94

y(0.0)=2
y(0.2)=y(0.0)+h*y'(0.0)
y(0.4)=y(0.2)+h*y'(0.2)
y(0.6)=y(0.4)+h*y'(0.4)
y(0.8)=y(0.6)+h*y'(0.6)
y(1.0)=y(0.8)+h*y'(0.8)

Problem 4:

Textbook: 8-21

(1+x^2)y'-xy=0,

y'=xy/(1+x^2),

Δx= 0.25 =h, 0.5*h= 0.125 1<=x<=2

x	y-RK	s1	y1	s2	y2	s3	y3	s4	y-exact	error(%)
1	2	1	2.125	1.055172	2.131897	1.058597	2.264649	1.104707	2	0
1.25	2.263844	1.104314	2.401883	1.142517	2.406658	1.144789	2.550041	1.176942	2.263846	-0.00012
1.5	2.549505	1.176694	2.696592	1.203629	2.699958	1.205132	2.850788	1.228032	2.54951	-0.0002
1.75	2.850432	1.227878	3.003916	1.247301	3.006344	1.248309	3.162509	1.265004	2.850439	-0.00024
2	3.162269								3.162278	-0.00027

y-exact=sqrt(2*(1+x*x))

error(%)=(y-RK - y-exact) / y-exact * 100%

y(1.0)=2.0

s1=f'(xi,yi)

s2=f'(xi+0.5h,yi+0.5h*s1), where y1=yi+0.5h*s1

s3=f'(xi+0.5h,yi+0.5h*s2), where y2=yi+0.5h*s2

s4=f'(xi+h,yi+h*s3), where y3=yi+h*s3

y_RK=yi+(h/6)(s1+2*s2+2*s3+s4)

Problem 5:

Textbook: 8-33

$$\frac{dy}{dx} + y - 1 = 0 \qquad \text{such that } y = 2 \text{ at } x = 0$$

$$\frac{dy}{dx} = 1 - y$$

Quadratic Model: $\hat{y} = b_0 + b_1 x + b_2 x^2$

Using the boundary condition

$$\hat{y} = 2 = b_0 + b_1(0) + b_2(0)^2$$

yields $b_0 = 2$. Thus, the quadratic model is

$$\hat{y} = 2 + b_1 x + b_2 x^2$$

and

$$\frac{d\hat{y}}{dx} = b_1 + 2b_2 x$$

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx} = b_1 + 2b_2 x - (1 - y) = b_1 + 2b_2 x + y - 1 = b_1 + 2b_2 x + (2 + b_1 x + b_2 x^2) - 1$$

$$= b_1(1 + x) + b_2(2x + x^2) + 1$$

$$\frac{\partial e}{\partial b_1} = 1 + x$$

$$\frac{\partial e}{\partial b_2} = 2x + x^2$$

Using Eq. 8-42 of the textbook, the following results can be obtained:

$$\int_0^x \left[b_1(1 + x) + b_2(2x + x^2) + 1 \right](1 + x)\ dx = 0$$

$$\int_0^x \left[b_1(1 + x) + b_2(2x + x^2) + 1 \right](2x + x^2)\ dx = 0$$

Using $x = 1$ as the upper limit, above two equations yield the following two normal equations:

$$2.3333b_1 + 2.2500b_2 + 1.5000 = 0$$

$$2.2500b_1 + 2.5333b_2 + 1.3333 = 0$$

The solution of the normal equations is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -0.94293 \\ 0.31117 \end{bmatrix}$$

Thus, the quadratic approximation is given by

$$\hat{y} = 2 - 0.94293x + 0.31117x^2$$

For $x = 0.2$, $\hat{y} = 2 - 0.94293(0.2) + 0.31117(0.2)^2 = 1.82386$

The following table summarizes the estimated values of the function $\hat{y}$ for the range of x = 0 to 1 in an increment of 0.2. The table also compares the computed values with the true values.

x	Estimated $y(x)$	True $y(x)$	error(%)
0	2	2	0.00000
0.2	1.823861	1.818731	0.28207
0.4	1.672615	1.670320	0.13741
0.6	1.546263	1.548812	0.16454
0.8	1.444805	1.449329	0.31216
1	1.368240	1.367879	0.02636

<u>Problem 6:</u>

Textbook: 8-35

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y - 1 = 0 \qquad\qquad \text{such that } y = 2 \text{ at } x = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y$$

Quadratic Model: $\hat{y} = b_0 + b_1 x + b_2 x^2$

Using the boundary condition

$$\hat{y} = 2 = b_0 + b_1(0) + b_2(0)^2$$

yields $b_0 = 2$. Thus, the quadratic model is

$$\hat{y} = 2 + b_1 x + b_2 x^2$$

and

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x} = b_1 + 2b_2 x$$

$$e = \frac{\mathrm{d}\hat{y}}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} = b_1 + 2b_2 x - (1-y) = b_1 + 2b_2 x + y - 1 = b_1 + 2b_2 x + (2 + b_1 x + b_2 x^2) - 1$$

$$= b_1(1+x) + b_2(2x + x^2) + 1$$

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_1} = x$$

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_2} = x^2$$

Using Eq. 8-57 of the textbook, the following results can be obtained:

$$\int_0^x \left[b_1(1+x) + b_2(2x+x^2) + 1\right](x)\ \mathrm{d}x = 0$$

$$\int_0^x \left[b_1(1+x) + b_2(2x+x^2) + 1\right](x^2)\ \mathrm{d}x = 0$$

Using $x = 1$ as the upper limit, above two equations yield the following two normal equations:

$$0.83333b_1 + 0.91667b_2 + 0.50000 = 0$$

$$0.58333b_1 + 0.70000b_2 + 0.33333 = 0$$

The solution of the normal equations is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -0.91435 \\ 0.28577 \end{bmatrix}$$

Thus, the quadratic approximation is given by

$$\hat{y} = 2 - 0.91435x + 0.28577x^2$$

For $x = 0.2$, $\hat{y} = 2 - 0.91435(0.2) + 0.28577(0.2)^2 = 1.82856$

The following table summarizes the estimated values of the function $\hat{y}$ for the range of x = 0 to 1 in an increment of 0.2. The table also compares the computed values with the true values.

x	Estimated $y(x)$	True $y(x)$	error(%)
0	2	2	0.00000
0.2	1.828561	1.818731	0.54049
0.4	1.679983	1.670320	0.57852
0.6	1.554267	1.548812	0.35224
0.8	1.451413	1.449329	0.14378
1	1.371420	1.367879	0.25884