Types of Interest

- **Simple Interest**

\[ I = Pni \]

- Interest is due at the end of the time period. For fractions of a time period, multiply the interest by the fraction.

\[
\begin{align*}
P &= \text{principal} & $1,000 \\
i &= \text{interest rate} & 0.12 \\
n &= \text{number of years or periods} & 1 \\
I &= \text{interest} & $120.00
\end{align*}
\]
Types of Interest

- Compound Interest: The interest of the interest.

  - A loan of $1,000 is made at an interest of 12% for 5 years. The interest is due at the end of each year with the principal is due at the end of the fifth year. The following table shows the resulting payment schedule:

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount at start of year</th>
<th>Interest at year end</th>
<th>Owed amount at year end</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
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</table>

Type of Interest
**Compound Interest (cont’d)**

A loan of $1,000 is made at an interest of 12% for 5 years. The principal and interest are due at the end of the fifth year. The following table shows the resulting payment schedule:

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount at start of year</th>
<th>Interest at year end</th>
<th>Owed amount at year end</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>
Cash Flow

- Cash flow over time: Upward arrow means positive flow, downward arrow means negative flow. There are two cash flows to each problem (borrower and lender flows).

- Net cash flow: The arithmetic sum of receipts (+) and disbursements (-) that occur at the same point in time.
**Notations**

- \( P \) = a present single amount of money
- \( F \) = a future single amount of money, after \( n \) periods of time
- \( A \) = a series of \( n \) equal payments
- \( i \) = the rate of interest per interest period (usually one year)
- \( n \) = the number of periods of time (usually years)

**Interest Formulas**

- **Single Payment Compound-Amount Factor (SPCAF)**

  \[
  F = P(1 + i)^n
  \]
  OR

  \[
  F = \left(\frac{F}{P}, i, n\right)
  \]
Interest Formulas

Example:
- Let the principle \( P = 1000 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.
- The future sum is therefore

\[
F = P(1 + i)^n = 1000(1 + 0.12)^4 = 1,573.5
\]

Interest Formulas

- Single Payment Present-Worth Factor (SPPWF)

\[
P = \frac{P}{(1+i)^n}
\]

OR

\[
P = \left( \frac{P}{F}, i, n \right)
\]
Interest Formulas

**Example 1:**

- Let the principle \( F = 1000 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.
- The present worth is therefore

\[
P = \frac{F}{(1+i)^n} = \frac{1000}{(1+0.12)^4} = 635.5
\]

**Example 2:**

- Let the principle \( F = 1573.5 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.
- The present worth is therefore

\[
P = \frac{F}{(1+i)^n} = \frac{1573.5}{(1+0.12)^4} = 1000.0
\]
Interest Formulas

Uniform (Equal payment) Series

Compound-Amount Factor (USCAF)

\[ F = A \left( \frac{(1+i)^n - 1}{i} \right) \]

OR

\[ F = \left( \frac{F}{A}, i, n \right) \]

Example:

– Let \( A = 100 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.
– The future value is therefore

\[ F = A \left( \frac{(1+i)^n - 1}{i} \right) = 100 \left( \frac{(1+0.12)^4 - 1}{0.12} \right) = 477.9 \]
Interest Formulas

Uniform (Equal payment) Series
Sinking-Fund Factor (USSFF)

\[ A = F \left( \frac{i}{(1+i)^n - 1} \right) \]

OR

\[ A = \left( \frac{A}{F}, i, n \right) \]

Example:

- Let the future value \( F = 1000 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.
- Therefore

\[ A = F \left( \frac{i}{(1+i)^n - 1} \right) = 1000 \left( \frac{0.12}{(1+0.12)^4 - 1} \right) = 209.2 \]
Interest Formulas

- Uniform (Equal payment) Series

Capital-Recovery Factor (USCRF)

\[ A = P \left( \frac{i(1+i)^n}{(1+i)^n-1} \right) \]

OR

\[ A = \left( \frac{A}{P}, i, n \right) \]

NOTE: This is the case of loans (mortgages)

Example:

- Let the present worth \( P = 1000 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.

- Therefore

\[
A = P \left( \frac{i(1+i)^n}{(1+i)^n-1} \right) = 1000 \left( \frac{0.12(1+0.12)^4}{(1+0.12)^4-1} \right) = 329.2
\]
### Interest Formulas

**Uniform (Equal payment) Series**

**Present-Worth Factor (USPWF)**

\[
P = A \left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)
\]

OR

\[
P = \left(\frac{P}{A}, i, n\right)
\]

---

**Example:**

- Let \( A = 100 \), the interest rate \( i = 12\% \), and the number of periods \( n = 4 \) years.
- Therefore

\[
P = A \left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) = 100 \left(\frac{(1+0.12)^4 - 1}{0.12(1+0.12)^4}\right) = 303.7
\]
**Interest Formulas**

**Uniform Gradient-Series Factor**
- The gradient \((G)\) is a value in the cash flow that starts with 0 at the end of year 1, \(G\) at the end of year 2, \(2G\) at the end of year 3, and so on to \((n-1)G\) at the end of year \(n\)

\[
P = G \left( \frac{1}{i} \frac{n}{(1+i)^n - 1} \right)
\]

**Example:**
- Let \(G = 100\), the interest rate \(i = 12\%\), and the number of periods \(n = 4\) years.
- Therefore

\[
A = G \left( \frac{1}{i} \frac{n}{(1+i)^n - 1} \right) = 100 \left( \frac{1}{0.12} - \frac{4}{(1+0.12)^4 - 1} \right) = 135.9
\]
Discrete and Continuous Compounding

- Nominal Interest Rate
  - It is expressed in annual basis
  - Financial institutions refer to this rate as the annual percentage rate or APR

- Effective Interest Rate
  - It is an interest rate that is compounded using a time period less than a year
  - The nominal interest rate in this case is the effective rate times the number of compounding periods in a year
  - It is referred to as nominal rate compounded at the period less than a year

Example:
- Effective rate is 1% per month
- Therefore
  Nominal Rate = 1% (12) = 12% compounded monthly
**Discrete and Continuous Compounding**

- **Relationship Between the Effective Interest Rate for any given time interval and the Nominal Interest Rate per Year**

- **Define**
  
  \[ r = \text{nominal interest rate per year} \]
  
  \[ i = \text{effective interest rate in the time interval} \]
  
  \[ l = \text{length of the time interval (in years)} \]
  
  \[ m = \text{reciprocal of the length of the compounding period (in years)} \]

\[
i = \left( 1 + \frac{r}{m} \right)^{lm} - 1
\]

- **The effective interest rate**

\[ r = \text{nominal interest rate per year} \]

\[ i = \text{effective interest rate in the time interval} \]

\[ l = \text{length of the time interval (in years)} \]

\[ m = \text{reciprocal of the length of the compounding period (in years)} \]
Discrete and Continuous Compounding

- If the interest is compounded only once in the time interval, then $l(m) = 1$, and

$$i = \frac{r}{m}$$

- $r =$ nominal interest rate per year
- $i =$ effective interest rate in the time interval
- $l =$ length of the time interval (in years)
- $m =$ reciprocal of the length of the compounding period (in years)

---

Discrete and Continuous Compounding

- To find the applicable effective interest rate for any time interval, the following equation can be used:

$$i = \left(1 + \frac{r}{m}\right)^c - 1$$

- $r =$ nominal interest rate per year
- $i =$ effective interest rate in the time interval
- $c =$ number of compounding periods in the time interval
- $m =$ reciprocal of the length of the compounding period (in years)
Continuous Compounding

- The limiting case for the effective rate is when compounding is performed an infinite times in a year, that is continuously. Using \( l = 1 \), the following limit produces the continuously compounded interest rate \( (i_a) \):

\[
i_a = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1
\]

Example 1

If the nominal rate of 12% is compounded monthly with time interval of one year, then \( c = 12 \), and

\[
i = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 0.1268 \text{ or } 12.68\% \text{ per year}
\]
Example 2

If the nominal rate of 18% is compounded weekly with time interval of one year, then:

c = 52, and

\[ i = \left( 1 + \frac{0.18}{52} \right)^{52} - 1 = 0.1968 \text{ or } 19.68\% \text{ per year} \]

Example 3

If the nominal rate of 14% is compounded monthly with time interval of six months, then:

c = 6, and

\[ i = \left( 1 + \frac{0.14}{12} \right)^{6} - 1 = 0.0721 \text{ or } 7.21\% \text{ per six months} \]
Discrete and Continuous Compounding

Example 4

If the nominal rate of 9% is compounded semiannually with time interval of two years, then
c = 4, and

\[ i = \left(1 + \frac{0.09}{2}\right)^4 - 1 = 0.1925 \text{ or } 19.25\% \text{ per two years} \]

Example 5

The effective interest rates corresponding to a nominal annual interest rate of 18% compounded annually, semiannually, quarterly, monthly, weekly, daily, and continuously are shown in the following Table:
Discrete and Continuous Compounding

Example 5 (cont’d)

<table>
<thead>
<tr>
<th>Compounding frequency</th>
<th>Number of periods per year</th>
<th>Effective interest rate per period Col. 3 = 18%/Col 2</th>
<th>Effective annual interest rate  ( i = \left(1 + \frac{0.18}{\text{Col 2}}\right) - 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>1</td>
<td>18%</td>
<td>18%</td>
</tr>
<tr>
<td>Semiannually</td>
<td>2</td>
<td>9</td>
<td>18.81</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4</td>
<td>4.5</td>
<td>19.2517</td>
</tr>
<tr>
<td>Monthly</td>
<td>12</td>
<td>1.5</td>
<td>19.5618</td>
</tr>
<tr>
<td>Weekly</td>
<td>52</td>
<td>0.3642</td>
<td>19.6843</td>
</tr>
<tr>
<td>Monthly</td>
<td>365</td>
<td>0.0493</td>
<td>19.7412</td>
</tr>
<tr>
<td>Continuously</td>
<td>( \infty )</td>
<td>0.00000</td>
<td>19.7217 = ( \exp(0.18) - 1 )</td>
</tr>
</tbody>
</table>

Comparing Interest Rates

- Since the effective interest rate represents the actual interest earned, this rate should be used to compare the benefits of various nominal rates of interest.
- For example, one might be confronted with the problem of determining whether it is more desirable to receive 16% compounded annually or 15% compounded monthly.
Discrete and Continuous Compounding

Comparing Interest Rates

- The effective rate of interest per year for 16% compounded annually is, of course, 16%
- However, for 15% compounded monthly, the effective annual rate is

\[
\left(1 + \frac{0.15}{12}\right)^{12} - 1 = 16.08\%
\]

- Thus, 15% compounded monthly yields an actual rate of interest that is higher than 16% compounded annually

Use of Interest Tables

- To aid in calculations using the relationships between \(P, F, A, n,\) and \(i,\) you can also find values for \(F/A, A/F, P/A,\) and \(A/P\) in tables for typical values for \(i\) and \(n\)
- The following table gives these factors for \(i = 23\) and
Use of Interest Tables

Example

- Let $A = 300$, the interest rate $i = 20\%$, and the number of periods $n = 5$ years.

- Using the table, then

\[
P(A, i, n) = \frac{P}{A} \times 20.5 = 300(2.9906) = 897.2
\]

\[
F(A, i, n) = \frac{F}{A} \times 20.5 = 300(7.4416) = 2,232.5
\]