CHAPTER

Duxbury
Thomson
Learning

Making Hard Decision

Third Edition

Theoretical Probability Models

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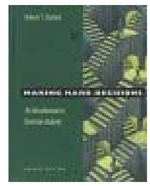

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ENCE 627 – Decision Analysis for Engineering

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Counting Sample Points

- Fundamental Principle of Counting
  - In many cases, a probability problem can be solved by counting the number of points in the sample space *S* without actually listing each elements.
  - In experiments that result in finite sample spaces, the process of identification, enumeration, and counting are essential for the purpose of determining the *probabilities* of some outcome of interest.

Counting Sample Points

- Multiplication Principle

1. If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.
2. In general, if there are n_k operations, then the n_k operation can be performed together in $n_1 n_2 n_3 \ldots\ldots n_k$

Counting Sample Points

- Example:
  - How many sample points are in the sample space when a pair of dice are thrown once?
  - The first die can land in any one of $n1 = 6$ ways. For each of these 6 ways the second die can also land in $n2 = 6$ ways. Therefore, a pair of dice can land in

Counting Sample Points

Example (cont'd)

SECOND DIE

FIRST DIE	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Sample space points = (6) (6) = 36 points

Counting Sample Points

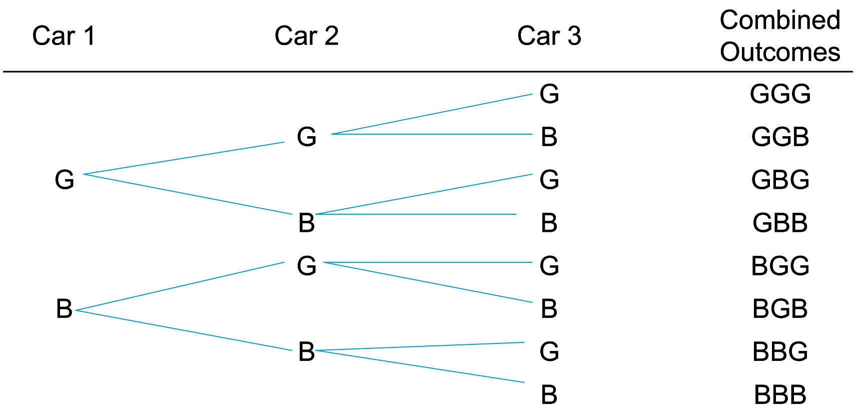
- Example: Three Cars
  - Assume that a car can only be in good (G) operating condition or bad (B) operating condition.
  - If there are three cars, the following situations are possible:

G	G	G	B	B	B	G	B
G	G	B	G	B	G	B	B
G	B	G	G	G	B	B	B

Sample space points = (2) (2) (2) = 8 events

Counting Sample Points

- <u>Example (cont'd):</u> Three Cars



Counting Sample Points

- Permutation

  The permutation of *r* elements from a set of *n* elements is the number of arrangements that can be made by selecting *r* elements out of the *n* elements:

  $$\boxed{P_{r,n} = \frac{n!}{(n-r)!} \qquad \text{for } 0 \le r \le n}$$

  The order of selection *<u>counts</u>* in determining these arrangements (**order matters**)

Counting Sample Points

- Combination

The combination of *r* elements from a set of *n* elements is number of arrangement that can be made by selecting *r* elements out of the *n* elements:

$$\boxed{C_{r,n} = \binom{n}{r} = \frac{n!}{(r!)(n-r)!} \qquad \text{for } 0 \le r \le n}$$

The order of selection <u>does not *counts*</u> in determining these arrangements (**order does not matter**)

Counting Sample Points

- <u>Example:</u> Permutations and Combinations

From a committee of 10 people:

a) In how many ways we can choose a chairperson, a vice chairperson, and a secretary, assuming that one person cannot hold more than one position?

b) In how many ways can we choose a subcommittee of 3 people?

Counting Sample Points

- Example: (cont'd)
  - Number of permutations:

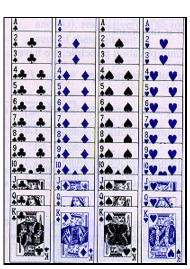
$$P_{r,n} = \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!} = 720 \text{ ways}$$

  - The number of combinations:

$$C_{r,n} = \binom{n}{r} = \frac{n!}{(r!)(n-r)!} = \frac{10!}{3!(10-3)!} = 120 \text{ ways}$$

Counting Sample Points

- Example: Standard 52-Card Deck



A) In drawing 5 cards from a 52-card deck without replacement, what is the probability of getting 5 spade? (note: order does not matter)

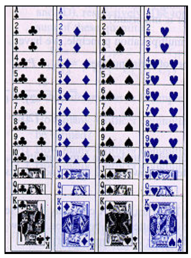
$n(S)$ = C5,52 $n(E)$ = C5,13

$$\mathrm{P}(E) = \frac{n(E)}{n(S)} = \frac{C_{5,13}}{C_{5,52}} = \frac{\dfrac{13!}{5!(13-5)!}}{\dfrac{52!}{5!(52-5)!}} = \frac{1287}{2598960} \approx 0.0005$$

Counting Sample Points

- Example: Standard 52-Card Deck

B) In drawing 5 cards from a 52-card deck without replacement, what is the probability of getting 2 kings and 3 queens?



$n(S) = C_{5,52}$ $n(E) = C_{2,4}\, C_{3,4}$

$$n(E) = C_{2,4}C_{3,4} = \frac{4!}{2!(4-2)!} \bullet \frac{4!}{3!(4-3)!} = 24$$

$$\mathrm{P}(E) = \frac{n(E)}{n(S)} = \frac{C_{2,4}C_{3,4}}{C_{5,52}} = \frac{24}{\frac{52!}{5!(52-5)!}} = \frac{1287}{2598960} \approx 0.000009$$

Counting Sample Points

- Example: Counting for Bridge Failure

Consider a bridge that is supported by three cables. The failure of interest is the failure of only two cables out of three cables since it results in failure of the bridge. What is the number of combinations of r = 2 out of n = 3 that can result in bridge failure?

$$C_{r,n} = \frac{n!}{r!(n-r)!} = \frac{3!}{2!(3-2)!} = \frac{(3)(2)(1)}{(2)(1)(1)} = 3$$

Counting Sample Points

- Example (cont'd): Counting for Bridge Failure

This number of combinations can be established by enumeration. The following events can be defined:

C_i = failure of cable *i*,where *i* =1, 2, and 3

The following events result in bridge failure:

$$C_1 \cap C_2 \qquad C_1 \cap C_3 \qquad C_2 \cap C_3$$

Counting Sample Points

- Example (cont'd): Counting for Bridge Failure

If we assume that the order of failure of the bridge is a factor, then the possible events become $C_1 \cap C_2 \quad C_1 \cap C_3 \quad C_2 \cap C_3$

$C_2 \cap C_1 \quad C_3 \cap C_1 \quad C_3 \cap C_2$

Therefore, the number of combinations in this case is six

$$P_{r,n} = \frac{n!}{(n-r!)} = \frac{3!}{(3-2)!} = \frac{(3)(2)(1)}{(1)} = 6$$

Counting Sample Points

- <u>Example (cont'd):</u> Counting for Bridge Failure

Now, if we assume that the bridge is supported by 20 cables, and the failure of 8 cables results in the failure of the bridge, what is the number of combinations that can result in bridge failure?

$$C_{r,n} = \frac{n!}{r!(n-r)!} = \frac{20!}{8!(20-8)!} = \frac{(20)(19)\ldots(13)(12!)}{(8)(7)(6)\ldots(1)(12!)} = 125970$$

Counting Sample Points

- <u>Example (cont'd):</u> Counting for Bridge Failure

For a real bridge, its failure can result from the failure of at least r = 8 out of n = 20. The number of combinations in this case is

$$\sum_{r=8}^{20} C_{r,n} = \frac{20!}{8!(12!)} + \frac{20!}{9!(11!)} + \frac{20!}{10!(10!)} + \ldots + \frac{20!}{20!(0!)}$$

Commonly Used Probability Distributions

- Any mathematical model satisfying the properties of PMF or PDF and CDF can be used to quantify uncertainties in a random variable.
- There are many different procedures to be discussed later for selecting a particular distribution for a random variable, and estimating its parameters.

Commonly Used Probability Distributions

- Many distributions are commonly used in the engineering profession to compute probability or reliability of events.
- Many computer programs and spreadsheets, such as MATLAB and EXCEL are used for probability calculations with various assumed ***theoretical*** distributions.

Common Discrete Probability Distributions

- A probability distribution function is expressed as a real-valued function of the random variable.
- The location, scale, and shape of the function are determined by its parameters.
- Distributions commonly have one to three parameters.

Common Discrete Probability Distributions

- These parameters take certain values that are specific for the problem being investigated.
- The parameters can be expressed in terms of the *mean*, *variance*, and *skewness*, but not necessarily in closed-form expressions

Common Discrete Probability Distributions

- Commonly Used Discrete Distributions
  - Bernoulli
  - Binomial
  - Geometric
  - Poisson
  - Other Distributions

Common Discrete Probability Distributions

- Bernoulli Trials and Binomial Distributions
  - If a coin is tossed, either a head occurs or it does not occur.
  - If a die is rolled, either a 3 shows or it does not show
  - If one is vaccinated for smallpox, either he or she contract smallpox or he or she does not.
  - A bridge failed or did not fail.

Common Discrete Probability Distributions

- Bernoulli Trials
  - What do all these situations have in common? All can be classified as experiments with *two possible outcomes*, each is the complement of the other.
  - An experiment for which there are only two possible outcomes, E or $\overline{E}$, is called a Bernoulli experiment or trial.

Common Discrete Probability Distributions

- Bernoulli Trials
  - In Bernoulli experiment or trial, it is customary to refer to one of the two outcomes as a ***success S*** and to the other as a ***failure F***.
  - If the probability of success is designated by $P(S) = p$, then the probability of failure is $P(F) = 1 - p = q$

Common Discrete Probability Distributions

- Bernoulli Distribution

The random variable X is defined as a mapping from the sample space $\{S, F\}$ for each trial of a Bernoulli sequence to the integer values {1, 0}. The probability function is given by

$$P_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1 - p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance are given by

$$\mu_X = p$$

$$\sigma_X^2 = p(1-p)$$

Common Discrete Probability Distributions

- Example: Roll of a Fair Die

If a fair die is rolled, what is the probability of 6 turning up? This can be viewed as a Bernoulli distribution by identifying a success with 6 turning up and a failure with any of the other numbers turning up. Therefore,

$$p = \frac{1}{6} \qquad \text{and} \qquad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Common Discrete Probability Distributions

- Example: Quality Assurance

  The quality assurance department in a structural-steel factory inspects every product coming off its production line. The product either fails or passes the inspection. Past experience indicates that the probability of failure (having a defective product) is 5%. Determine the average percent of the products that will pass the inspection. What are its variance and coefficient of variation?

Common Discrete Probability Distributions

- Example (cont'd): Quality Assurance
  - The average percent of the product that will pass the inspection is

$$\mu_X = E(X) = p = 1 - 0.05 = 0.95 = 95\%$$

  - Its variance and coefficient of variation (*COV*) are

$$\text{Var}(X) = p(1-p) = 0.95(1-0.95) = 0.0475$$

and

$$COV(X) = \frac{\sqrt{\text{Var}(X)}}{E(X)} = \frac{\sqrt{0.0457}}{0.95} = 0.229$$

Common Discrete Probability Distributions

- Bernoulli Trials
  - Suppose a Bernoulli trial is repeated a number of times. It becomes of interest to try to determine the probability of a given number of successes out of the given number of trials.
  - For example, one might be interested in the probability of obtaining exactly three 5's in six rolls of a fair die or the probability that 8 people will not catch flu out of 10 who have inoculated.

Common Discrete Probability Distributions

- Bernoulli Trials

Suppose a Bernoulli trial is repeated ***five*** times so that each trial is completely ***independent*** of any other and *p* is the probability of success on each trial. Then the probability of the outcome *SSFFS would be*

$$\begin{aligned}\mathrm{P}(SSFFS) &= P(S)P(S)P(F)P(F)P(S)\\ &= ppqqp = p^3q^2\\ &= p^3(1-p)^2\end{aligned}$$

Common Discrete Probability Distributions

- Bernoulli Trials

  A sequence of experiment is called a **sequence of Bernoulli trials**, or a **binomial experiment**, if

  1. Only two outcome are possible on each trial.
  2. The probability of success *p* for each trial is constant.
  3. All trials are independent

Common Discrete Probability Distributions

- Example A: Roll of Fair Die Five Times

  If a fair die is rolled five times and a success is identified in a single roll with 1 turning up, what is the probability of the sequence *SFFSS* occurring?

$$p = \frac{1}{6} \qquad q = 1 - p = \frac{5}{6}$$

$$\mathrm{P}(SFFSS) = pqqpp = p^3 q^2$$

$$= p^3(1-p)^2 = \left(\frac{1}{6}\right)^3 \left(1 - \frac{1}{6}\right)^2 = 0.003$$

Common Discrete Probability Distributions

- Example B: Roll of Fair Die Five Times

  If a fair die is rolled five times and a success is identified in a single roll with 1 turning up, what is the probability of the sequence *FSSSF* occurring?

$$p = \frac{1}{6} \qquad q = 1 - p = \frac{5}{6}$$

$$\mathrm{P}(FSSSF) = qpppq = p^3 q^2$$

$$= p^3 (1-p)^2 = \left(\frac{1}{6}\right)^3 \left(1 - \frac{1}{6}\right)^2 = 0.003$$

Common Discrete Probability Distributions

- Example C: Roll of Fair Die Five Times

  If a fair die is rolled five times and a success is identified in a single roll with 1 turning up, what is the probability of obtaining exactly three 1's?

Notice how this problem differs from Example B. In that example we looked at one way three 1's can occur. Then in Example A, we saw another way.

Common Discrete Probability Distributions

- Example C: Roll of Fair Die Five Times

Thus exactly three 1's may occur in the following sequences (among others):

SFFSS *FSSSF*

The probability in Example A and B of each sequence occurring is the same, namely,

$$\mathrm{P}(FSSSF) = \mathrm{P}(SFFSS) = 0.003$$

$$\text{where } p = \frac{1}{6} \qquad q = 1 - p = \frac{5}{6}$$

Common Discrete Probability Distributions

- Example C: Roll of Fair Die Five Times

How many more sequence will produce exactly three 1's? To answer this question think of the number of ways the following five blank positions can be filled with three *S*'s and two *F*'s:

$$b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5$$
$$\square \quad \square \quad \square \quad \square \quad \square$$

Common Discrete Probability Distributions

- Example C: Roll of Fair Die Five Times

$$\begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \\ \square & \square & \square & \square & \square \end{array}$$

  - A given sequence is determined once the *S*'s are located. Thus we are interested in the number of ways three blank positions can be selected for the *S*'s out of the five available blank positions b_1, b_2, b_3, b_4, and b_5.
  - This problem should sound familiar – it is just the problem of finding the number of combinations of 5 objects taken 3 at a time.

Common Discrete Probability Distributions

- Example C: Roll of Fair Die Five Times
  - That is, $C_{3,5}$. Thus the number of different sequences of successes and failures that produce exactly three successes (exactly three 1's) is

$$C_{3,5} = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{120}{6(2)} = \frac{120}{12} = 10$$

  - The probability of each sequence is the same, that is

$$p^3q^2 = p^3(1-p)^2 = \left(\frac{1}{6}\right)^3\left(1-\frac{1}{6}\right)^2 = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

Common Discrete Probability Distributions

- Example C: Roll of Fair Die Five Times
  - Since the probability of each sequence is the same (0.003) and there are 10 mutually exclusive sequences that produce exactly three 1's, we have

$$\mathrm{P}(\text{Exactly three successes}) = C_{3,5}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

$$= \frac{5!}{3!(5-2)!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

$$= (10)\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = 0.032$$

Common Discrete Probability Distributions

- Binomial Formula (Brief Review)

$$\boxed{(a+b)^n = C_{0,n}a^n + C_{1,n}a^{n-1}b + C_{2,n}a^{n-2}b^2 + \ldots + C_{n,n}b^n}$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Common Discrete Probability Distributions

- Binomial Distribution Versus Binomial Formula
  - The probabilistic characteristic of the car problem considered previously can be described by the binomial distribution.
  - Since three cars are involved, $n = 3$.
  - Also, the probability of each car being good is 0.9 or $p = 0.9$
  - The binomial coefficients when $X = 0, 1, 2,$ and 3 can be shown to be 1, 3, 3, and 1, respectively

Common Discrete Probability Distributions

- Binomial Distribution VS Binomial Formula
  - Three Cars Example
    - Let the random variable X represent the number of successes in three trials 0, 1, 2, or 3. We are interested in the probability distribution for this random variable. Which outcomes of an experiment consisting of a sequence of three Bernoulli trials lead to the random values 0, 1, 2, and 3, and what are the probabilities associated with these values? The following table answer these questions:

Common Discrete Probability Distributions

Three Cars Example

Outcome		Probability of Simple Event	Frequency	x successes in 3 trials	P(X = x)
BBB	*FFF*	$qqq = q^3$	1	0	q^3
BBG	*FFS*	$qqp = q^2p$	3	1	$3q^2p$
BGB	*FSF*	$qpq = q^2p$			
GBB	*SFF*	$pqq = q^2p$			
BGG	*FSS*	$qpp = qp^2$	3	2	$3qp^2$
GBG	*SFS*	$pqp = qp^2$			
GGB	*SSF*	$ppq = qp^2$			
GGG	*SSS*	$ppp = p^3$	1	3	p^3

Common Discrete Probability Distributions

- Binomial Distribution VS Binomial Formula
  - The terms in the last column of the previous table are the terms in the binomial expansion of $(q + p)^3$.
  - The last two columns in the table provide a probability distribution for the random variable X.

$$
\begin{aligned}
1 = 1^3 = (q+p)^3 &= C_{0,3}q^3 + C_{1,3}q^2p + C_{2,3}qp^2 + C_{3,3}p^3 \\
&= q^3 + 3q^2p + 3qp^2 + p^3 \\
&= \mathrm{P}(X=0) + \mathrm{P}(X=1) + \mathrm{P}(X=2) + \mathrm{P}(X=3)
\end{aligned}
$$

Common Discrete Probability Distributions

- Binomial Distribution
  - The underlying random variable X for this distribution represents the number of successes in N Bernoulli trials. The probability mass function is given by

$$P_X(x) = \begin{cases} \binom{N}{x} p^x (1-p)^{N-x} & \text{for } x = 0,1,2,\ldots,N \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance are given by

$$\mu_X = Np$$

$$\sigma_X^2 = Np(1-p)$$

Common Discrete Probability Distributions

- Characteristic of Binomial Distribution
  1. The distribution is based on N Bernoulli trials with only two possible outcomes.
  2. The N trials are *independent* of each other.
  3. The probabilities of the outcomes remain constant at p and $(1 – p)$ for each trial

Common Discrete Probability Distributions

$$P_X(x)=\begin{cases}\binom{N}{x}p^x(1-p)^{N-x} & \text{for } x=0,1,2,\ldots,N\\ 0 & \text{otherwise}\end{cases}$$

- Example: Rolling of a Die

If a fair die is rolled five times. What is the probability of rolling: (a) exactly two 3's? and (b) at least 3's?

(a)
$$\begin{aligned}P(x=2)&=\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3\\&=\frac{5!}{2!(5-2)!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3\\&=0.161\end{aligned}$$

Common Discrete Probability Distributions

$$P_X(x)=\begin{cases}\binom{N}{x}p^x(1-p)^{N-x} & \text{for } x=0,1,2,\ldots,N\\ 0 & \text{otherwise}\end{cases}$$

- Example: Rolling of a Die

(b)
$$P(x\geq 2)=P(x=2)+P(x=3)+P(x=4)+P(x=5)$$

It is easier to compute the probability of the complement of this event, P(x<2), and use

$$\begin{aligned}P(x\geq 2)&=1-P(x<2)=1-P(x=0)-P(x=1)\\&=1-\binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5-\binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4\\&=1-\frac{5!}{0!5!}\left(\frac{5}{6}\right)^5-\frac{5!}{1!4!}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4\\&=1-0.402-0.402=0.196\end{aligned}$$

Common Discrete Probability Distributions

- Geometric Distribution
  - The underlying random variable X for this distribution represents the number of Bernoulli trials that are required to achieve the first success. The probability mass function is given by

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & \text{for } x = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance are given by

$$\mu_X = \frac{1}{p} \qquad \sigma_X^2 = \frac{1-p}{p^2}$$

Common Discrete Probability Distributions

- Example: Traffic Accidents

Based on previous accident records, the probability of being in a fatal traffic accident is on the average $1.8X10^{-3}$ per 1000 miles of travel. What is the probability of being in a fatal accident for the first time at 10,000 and 100,000 miles of travel?

$$P_X(10{,}000) = 1.8\times10^{-3}\left(1-1.8\times10^{-3}\right)^{10-1} = 1.77\times10^{-3}$$

$$P_X(100{,}000) = 1.8\times10^{-3}\left(1-1.8\times10^{-3}\right)^{100-1} = 1.51\times10^{-3}$$

Common Discrete Probability Distributions

- Example: Defective Items

In certain manufacturing process it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Using x = 5, and p = 0.01, we have

$$P_X(X) = p(1-p)^{x-1}$$
$$= (0.01)(1-0.01)^{5-1}$$
$$= (0.01)(0.99)^4 = 0.0096$$

Common Discrete Probability Distributions

- Poisson Distribution
  - This is another important distribution used frequently in engineering to evaluate the ***risk of damage***.
  - It is used in engineering problems that deal with the occurrence of some random event in the continuous dimension of time or space.

Common Discrete Probability Distributions

- The number of occurrences of natural hazard, such as earthquakes, tornadoes, or hurricanes, in some time interval, such as one year, can be considered as random variable with Poisson distribution.
- In these examples, the number of occurrences in the time interval is the random variable. Therefore, the random variable is discrete, whereas its reference space, the time interval is continuous.

Common Discrete Probability Distributions

- This distribution is considered the limiting case of the binomial distribution by dividing the reference space (time t) into non-overlapping interval of size Δt.
- The occurrence of an event (i.e., a natural hazard) in each interval is considered to constitute a Bernoulli sequence.
- By considering the limiting case where the size Δt approaches zero, the binomial distribution becomes *Poisson distribution*.

Common Discrete Probability Distributions

- Poisson Distribution
  - The underlying random variable of this distribution is denoted by X_t, which represents the number of occurrences of an event of interest, and t = time interval. The PMF is

$$P_{X_t}(x) = \begin{cases} \dfrac{(\lambda t)^x e^{-\lambda t}}{x!} & \text{for } x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance are given by

$$\mu_X = \lambda t$$

$$\sigma_X^2 = \lambda t$$

Common Discrete Probability Distributions

- <u>Example: Tornadoes</u>

  From the records of the past 50 years, it is observed that tornadoes occur in a particular area an average of two times a year. In this case, λ = 2/year. The probability of no tornadoes in the next year (i.e., x = 0, and t = 1 year) can be computed as follows:

Common Discrete Probability Distributions

- Example (cont'd): Tornadoes

$$P(\text{no tornado next year}) = \frac{(\lambda t)^x e^{-\lambda t}}{x!} = \frac{(2\times 1)^0 e^{-2\times 1}}{0!} = 0.135$$

$$P(\text{exactly 2 tornadoes next year}) = \frac{(2\times 1)^0 e^{-2\times 1}}{2!} = 0.271$$

$$P(\text{no tonadoes in next 50 years}) = \frac{(2\times 50)^0 e^{-2\times 50}}{0!} = 3.72\times 10^{-44}$$

Common Discrete Probability Distributions

- Negative Binomial Distribution
  - Consider an experiment in which the properties are the same as those listed for a *binomial experiment*, with the exception that the trials will be repeated until a *fixed* number of successes occur.
  - Therefore, instead of finding the probability of x successes in N trials, where *N* is fixed, the interest now is in the probability that the k^{th} successes occurs on the x^{th} trial. Experiments of this type are called NBD.

Common Discrete Probability Distributions

- Negative Binomial Distribution (NBD)
  - If repeated independent trials can result in a success with probability p, then the probability distribution of the random variable X, the number of trial on which the k^{th} success occurs, is given by

$$P_{X_t}(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & \text{for } x = k, k+1, k+2, \ldots. \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance are given by

$$\mu_X = \frac{k}{p} \qquad \sigma_X^2 = \frac{k(1-p)}{p^2}$$

Common Discrete Probability Distributions

- Example: Tossing Three Coins

Find the probability that a person tossing three coins will get either all heads or tails for the second time on the fifth toss.

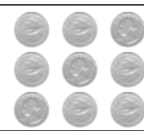
With x = 5, k =2, p =2/8 = 1/4,

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} = \binom{5-1}{2-1}\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{5-2}$$

$$= \binom{4}{1}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 = \frac{27}{256} = \boxed{0.1054}$$

Common Discrete Probability Distributions

- <u>Example (cont'd):</u> Tossing Three Coins

Outcome		Number of Heads	Frequency
TTT		0	1
(TTH), (THT), and (HTT)		1	3
(THH), (HTH), and (HHT)		2	3
(HHH)		3	1

Common Discrete Probability Distributions

- <u>Example:</u> Radio Tower

A radio transmission tower is designed for a 50-year wind. The probability of encountering the 50-year wind in any one year is p = 0.02.

a) What is the probability that the design wind velocity will be exceeded for the first time on the fifth year after completion of the structure?

b) What is the probability that a second 50-year wind will occur exactly on the fifth year after completion of the structure?

Common Discrete Probability Distributions

- Example (cont'd): Radio Tower

  a) Note: this is a geometric distribution:

$$P_X(x) = P(x=5) = p(1-p)^{x-1}$$
$$= (0.02)(0.98)^{5-1} = 0.018$$

  b) Note: this a Negative Binomial Distribution

$$P_X(x) = P_X(x=5) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$
$$= \binom{5-1}{2-1} (0.02)^2 (1-0.02)^{5-2}$$
$$= \binom{4}{1} (0.02)^2 (0.98)^3 = 0.0015$$

Common Discrete Probability Distributions

- Special Case of Negative Binomial Distribution (NBD)
  - When k = 1, we get a probability distribution for the number of trials required for a single success. An example would be the tossing of a coin until a head occurs.
  - We might be interested in the probability that the first head occurs on the fourth toss.
  - The NBD reduces to the special case of ***Geometric Distribution***, $P_X(x) = p(1-p)^{x-1}$

Common Discrete Probability Distributions

- Hypergeometric Distribution

The probability distribution of the hypergeometric random variable X, the number of successes in a random sample size n selected from N items of which D are labeled success and $N - D$ labeled failure is

$$P_{X_t}(x) = \begin{cases} \dfrac{\dbinom{D}{x}\dbinom{N-D}{n-x}}{\dbinom{N}{n}} & \text{for } x = 0,1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance are given by

$$\mu_X = n\frac{D}{N} \qquad \sigma_X^2 = n\frac{D}{N}\frac{N-n}{N-1}\left(1-\frac{D}{N}\right)$$

Common Discrete Probability Distributions

- Example: Hypergeometric Distribution
  - If one wishes to find the probability of observing 3 red cards in 5 draws from an ordinary deck of 52 playing cards, the *binomial distribution* does not apply unless each card is replaced and the deck reshuffled before the next drawing is made.
  - To solve the problem of sampling without replacement, let us restate the problem.

Common Discrete Probability Distributions

- Example: Hypergeometric Distribution
  - If 5 cards are drawn at random, one is interested in the probability of selecting 3 red cards from 26 available and 2 black cards from 26 black cards available in the deck.

There are $\binom{26}{3}$ ways of selecting 3 red cards and for each of these ways we can choose 2 black cards in $\binom{26}{2}$ ways.

Common Discrete Probability Distributions

- Example: Hypergeometric Distribution
  - Therefore, the total number of ways to select 3 red and 2 black cards in 5 draws is

$$\binom{26}{3}\binom{26}{2}$$

  - The total number of ways to select any 5 cards from the 52 that are available is

$$\binom{52}{5}$$

Common Discrete Probability Distributions

- Example: Hypergeometric Distribution
  - Hence the probability of selecting 5 cards without replacement of which 3 are red and 2 are black is given by

$$P_X(x) = \frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = \frac{(26!/3!23!)(26!/2!24!)}{(52!/5!47!)} = \boxed{0.3251}$$

  - In general, we are interested in the probability of selecting x successes from the D items labeled success and $n-x$ failures from $N-k$ items labeled failures when a random sample of size n is selected from N items.
  - This is known as a ***hypergeometric* experiment**

Common Continuous Probability Distributions

- Continuous distributions:
  - Uniform
  - Normal
  - Lognormal
  - Exponential
  - Other Continuous Probability Distributions
    - Chi-square, Student-*t* and *F* distributions
    - Extreme Value Distributions
    - Others

Common Continuous Probability Distributions

- Uniform Distribution
  - The probability density function (PDF) for the uniform distribution of a random variable X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where $a < b$. The mean and variance are given by

$$\mu_X = \frac{a+b}{2} \qquad \sigma_X^2 = \frac{(b-a)^2}{12}$$

Common Continuous Probability Distributions

- Uniform Distribution
  - The cumulative distribution function (CDF) for the uniform distribution of a random variable X is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x \le a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for } x \ge b \end{cases}$$

where $a < b$. The mean and variance are given by

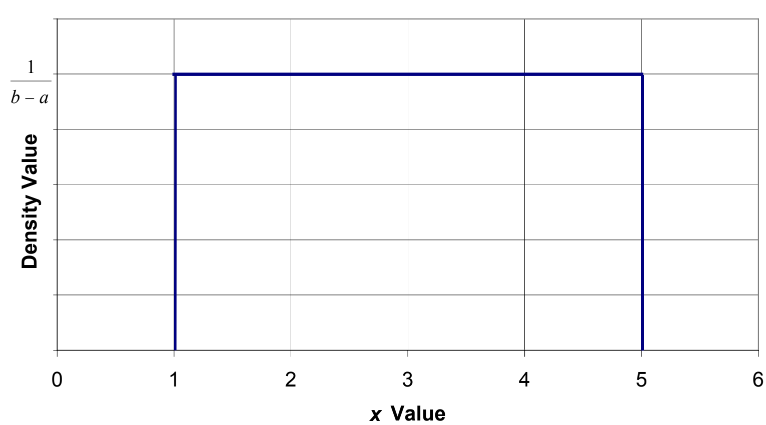
$$\mu_X = \frac{a+b}{2} \qquad \sigma_X^2 = \frac{(b-a)^2}{12}$$

Common Continuous Probability Distributions

- Uniform Distribution
  - The uniform distribution is very important for performing random number generation in simulation as will be described later in Ch. 11.
  - Due to its simplicity, it can be easily shown that its mean value and variance as given by the above equations, respectively, correspond to centroidal distance and centroidal moment of inertia with respect to a vertical axis of the area under the PDF.

Common Continuous Probability Distributions

Probability Density Function of the Uniform Distribution



Common Continuous Probability Distributions

- Example: Concrete Strength

Based on experience, a structural engineer assesses the strength of concrete in existing bridge to be in the range 3000 to 4000 psi. Find the mean, variance, standard deviation of strength of the concrete. What is the probability that the strength of concrete X is larger than 3600 psi?

Here we have a = 3000 psi and b = 4000 psi

Common Continuous Probability Distributions

- Example (cont'd): Concrete Strength

$$\text{Mean} = \mu_X = \frac{a+b}{2} = \frac{3000+4000}{2} = 3500 \text{ psi}$$

$$\text{Variance} = \sigma_X^2 = \frac{(b-a)^2}{12} = \frac{(4000-3000)^2}{12} = 83{,}333 \text{ psi}^2$$

$$\text{Standard Deviation} = \sqrt{83333} = 288.7 \text{ psi}$$

$$\begin{aligned}\text{P(strength of concrete larger than 3600 psi)} &= \text{P}(X > 3600) \\ &= 1 - F_X(3600) \\ &= 1 - \frac{x-a}{b-a} \\ &= 1 - \frac{3600-3000}{4000-3000} = 0.4\end{aligned}$$

Common Continuous Probability Distributions

- Normal (Gaussian) Distribution
  - The probability density function (PDF) for the normal distribution of a random variable X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]} \qquad -\infty < x < +\infty$$

It is common to use the notation $X \sim N(\mu, \sigma^2)$.

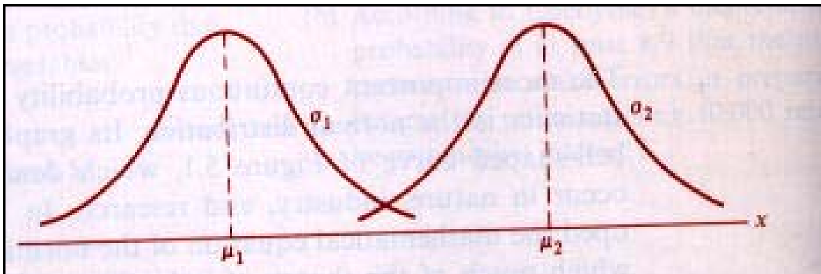
The notation states that X is normally distributed with a mean value μ and variance σ^2

Common Continuous Probability Distributions

- Properties of Normal Distribution
  1. $f_X(x)$ approaches 0 as x approaches either $-\infty$ or $+\infty$
  2. $f_X(a + \mu) = f_X(-a + \mu)$ for any a, *i.e.*, symmetric PDF about the mean.
  3. The maximum value of $f_X(x)$ (the mode) occurs at $x = \mu$.
  4. The inflection points of the density function occurs at $x = \mu \pm \sigma$.
  5. The density function has an overall bell shape
  6. The mean value μ and variance σ^2 are the parameters of the distribution.

Common Continuous Probability Distributions

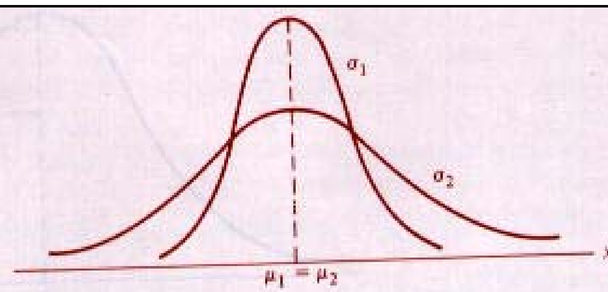
- Normal Distribution



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$.

Common Continuous Probability Distributions

- Normal Distribution



Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.

Common Continuous Probability Distributions

- Normal (Gaussian) Distribution
  - The cumulative distribution function (CDF) for the normal distribution of a random variable X is given by

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]} dx$$

It is common to use the notation $X \sim \mathrm{N}(\mu, \sigma^2)$.

The notation states that X is normally distributed with a mean value μ and variance σ^2

Common Continuous Probability Distributions

- Transformation of Normal Distribution
  - The evaluation of the integral of the previous equation requires numerical methods for each pair (μ, σ^2).
  - This difficulty can be avoided by performing a transformation that result in a ***standard normal distribution*** with a mean μ = 0 and variance σ^2 =1 denoted as $Z \sim \mathrm{N}(0,1)$
  - Numerical integration can be used to determine the cumulative distribution function of the standard normal distribution.

Common Continuous Probability Distributions

- Transformation of Normal Distribution
  - By using the transformation between the normal distribution $X \sim N(\mu, \sigma^2)$ and the standard normal distribution $Z \sim N(0,1)$, and the integration results for the standard normal, the cumulative distribution function for the normal distribution can be evaluated using the following transformation:

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

Common Continuous Probability Distributions

- Standard Normal Distribution
  - The density function and the cumulative distribution function of the standard normal given, respectively as

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

where $\varphi(z)$ = special notation for PDF of standard normal

$\Phi(z)$ = special notation for CDF of standard normal

Common Continuous Probability Distributions

■ Standard Normal Distribution

- The results of the integral $\Phi(z)$ are usually provided in tables (e.g., Appendix of Textbook).
- Negative z values can be obtained using the symmetry property of the normal distribution

$$\Phi(-z) = 1 - \Phi(z)$$

- The table can also be used to determine the inverse Φ^{-1} of the Φ . For specified values that are less than 0.5, the table can be used with

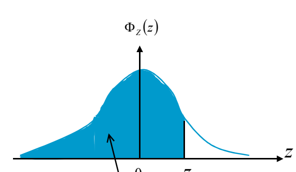
$$z = \Phi^{-1}(p) = -\Phi^{-1}(1 - |p|) \quad \text{for } p < 0.5$$

Common Continuous Probability Distributions

■ Sample Table of Standard Normal

$\mu = 0$
$\sigma = 1$

z	Φ(z)	z	Φ(z)	z	Φ(z)
0	0.5	0.2	0.57926	0.4	0.655422
0.01	0.503989	0.21	0.583166	0.41	0.659097
0.02	0.507978	0.22	0.587064	0.42	0.662757
0.03	0.511967	0.23	0.590954	0.43	0.666402
0.04	0.515953	0.24	0.594835	0.44	0.670031
0.05	0.519939	0.25	0.598706	0.45	0.673645
0.06	0.523922	0.26	0.602568	0.46	0.677242
0.07	0.527903	0.27	0.60642	0.47	0.680822
0.08	0.531881	0.28	0.610261	0.48	0.684386
0.09	0.535856	0.29	0.614092	0.49	0.687933
0.1	0.539828	0.3	0.617911	0.5	0.691462
0.11	0.543795	0.31	0.621719	0.51	0.694974
0.12	0.547758	0.32	0.625516	0.52	0.698468
0.13	0.551717	0.33	0.6293	0.53	0.701944
0.14	0.55567	0.34	0.633072	0.54	0.705402
0.15	0.559618	0.35	0.636831	0.55	0.70884
0.16	0.563559	0.36	0.640576	0.56	0.71226
0.17	0.567495	0.37	0.644309	0.57	0.715661
0.18	0.571424	0.38	0.648027	0.58	0.719043
0.19	0.575345	0.39	0.651732	0.59	0.722405



$P(Z \le z) = \Phi(z) = \text{shaded area}$

Common Continuous Probability Distributions

- Transformation to Standard Normal Distribution

$$P(X \le x) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]} dx$$

Changing the variable,

$$P(X \le x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma \, dz$$

$$P(X \le x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

It can be shown that

$$P(a \le X \le b) = F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Common Continuous Probability Distributions

- <u>Example:</u> Concrete Strength

The structural engineer of the previous example decided to use a normal distribution to model the strength of concrete. The mean and standard deviation are same as before, i.e., 3500 psi and 288.7 psi, respectively. What is the probability that the concrete strength is larger than 3600 psi?

$$\mu = 3500 \text{ psi} \qquad \text{and} \qquad \sigma = 288.7 \text{ psi}$$

Common Continuous Probability Distributions

- Example (cont'd): Concrete Strength

$$P(X > 3600) = 1 - P(X \leq 3600) = 1 - \Phi\left[\frac{3600 - 3500}{288.7}\right]$$
$$= 1 - \Phi(0.3464)$$

Using linear interpolation in the following table:

z	$\Phi(z)$
0.34	0.633072
0.3464	$\Phi(z)$
0.35	0.636831

$$\rightarrow \quad \frac{0.3464 - 0.34}{0.35 - 0.34} = \frac{\Phi(z) - 0.633072}{0.636831 - 0.633072}$$

Common Continuous Probability Distributions

- Example (cont'd): Concrete Strength

$$\frac{0.3464 - 0.34}{0.35 - 0.34} = \frac{\Phi(z) - 0.633072}{0.636831 - 0.633072}$$

$$\therefore \Phi(z) = \Phi(0.3464) = 0.635478$$

Therefore,

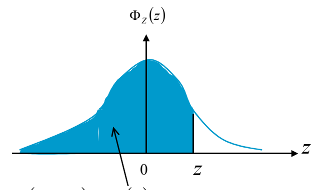
$$P(X > 3600) = 1 - 0.635478 = 0.364522$$

Common Continuous Probability Distributions

- Example (cont'd): Concrete Strength

$\mu = 0$
$\sigma = 1$

z	Φ(z)	z	Φ(z)	z	Φ(z)
0	0.5	0.2	0.57926	0.4	0.655422
0.01	0.503989	0.21	0.583166	0.41	0.659097
0.02	0.507978	0.22	0.587064	0.42	0.662757
0.03	0.511967	0.23	0.590954	0.43	0.666402
0.04	0.515953	0.24	0.594835	0.44	0.670031
0.05	0.519939	0.25	0.598706	0.45	0.673645
0.06	0.523922	0.26	0.602568	0.46	0.677242
0.07	0.527903	0.27	0.60642	0.47	0.680822
0.08	0.531881	0.28	0.610261	0.48	0.684386
0.09	0.535856	0.29	0.614092	0.49	0.687933
0.1	0.539828	0.3	0.617911	0.5	0.691462
0.11	0.543795	0.31	0.621719	0.51	0.694974
0.12	0.547758	0.32	0.625516	0.52	0.698468
0.13	0.551717	0.33	0.6293	0.53	0.701944
0.14	0.55567	0.34	0.633072	0.54	0.705402
0.15	0.559618	0.35	0.636831	0.55	0.70884
0.16	0.563559	0.36	0.640576	0.56	0.71226
0.17	0.567495	0.37	0.644309	0.57	0.715661
0.18	0.571424	0.38	0.648027	0.58	0.719043
0.19	0.575345	0.39	0.651732	0.59	0.722405



$P(Z \le z) = \Phi(z) = \text{shaded area}$

Common Continuous Probability Distributions

- Useful Properties of Normal Distribution

  1. The addition of n normally distributed random variables $X_1, X_2, \ldots, X_n$ is a normal distribution as follows:

$$Y = X_1 + X_2 + X_3 + \ldots + X_n$$

The mean of Y is

$$\mu_Y = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} + \ldots + \mu_{X_n}$$

Common Continuous Probability Distributions

The variance of Y is

$$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 + \ldots + \sigma_{X_n}^2$$

2. Central limit theorem: Informally stated, the addition of a number of individual random variables, without a dominating distribution type, approaches a normal distribution as the number of the random variables approaches infinity. The result is valid regardless of the underlying distribution types of the random variables.

Common Continuous Probability Distributions

- Example: Modulus of Elasticity

The randomness in the modulus of elasticity (or Young's modulus) E can be described by a normal random variable. If the mean and standard deviation were estimated to be 29,567 ksi and 1,507 ksi, respectively,

1. What is the probability of E having a value between 28,000 ksi and 29,500 ksi?
2. The commonly used Young's modulus E for steel is 29,000 ksi. What is the probability of E being less than the design value, that is $E \leq 29{,}000$ ksi?
3. What is the probability that E is at least 29,000 ksi?
4. What is the value of E corresponding to 10-percentile?

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

$$\mu = 29{,}576 \text{ ksi} \quad \text{and} \quad \sigma = 1{,}507 \text{ ksi}$$

1.

$$\begin{aligned} \mathrm{P}(28{,}000 < E \le 29{,}500) &= \Phi\left[\frac{b-\mu}{\sigma}\right] - \Phi\left[\frac{a-\mu}{\sigma}\right] \\ &= \Phi\left[\frac{29{,}000-29{,}576}{1{,}507}\right] - \Phi\left[\frac{28{,}000-29{,}576}{1{,}507}\right] \\ &= \Phi(-0.05) - \Phi(-1.05) = [1-\Phi(0.05)] - [1-\Phi(1.05)] \\ &= (1\text{-}0.51994)\text{-}(1\text{-}0.85314) = 0.33320 \end{aligned}$$

2.

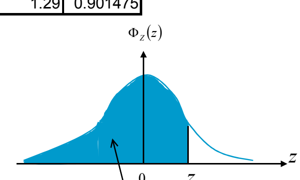
$$\begin{aligned} \mathrm{P}(E \le 29{,}000) &= \Phi\left(\frac{29{,}000-29576}{1{,}507}\right) = \Phi(-0.38) \\ &= 1-\Phi(0.38) = 1-0.64803 = 0.35197 \end{aligned}$$

Common Continuous Probability Distributions

- Sample Table of Standard Normal

$\mu = 0$
$\sigma = 1$

z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)
0	0.5	0.2	0.57926	1	0.841345	1.2	0.88493
0.01	0.503989	0.21	0.583166	1.01	0.843752	1.21	0.88686
0.02	0.507978	0.22	0.587064	1.02	0.846136	1.22	0.888767
0.03	0.511967	0.23	0.590954	1.03	0.848495	1.23	0.890651
0.04	0.515953	0.24	0.594835	1.04	0.85083	1.24	0.892512
0.05	0.519939	0.25	0.598706	**1.05**	**0.853141**	**1.25**	**0.89435**
0.06	0.523922	0.26	0.602568	1.06	0.855428	1.26	0.896165
0.07	0.527903	0.27	0.60642	1.07	0.85769	1.27	0.897958
0.08	0.531881	0.28	0.610261	1.08	0.859929	1.28	0.899727
0.09	0.535856	0.29	0.614092	1.09	0.862143	1.29	0.901475
0.1	0.539828	0.3	0.617911	1.1	0.864334		
0.11	0.543795	0.31	0.621719	1.11	0.8665		
0.12	0.547758	0.32	0.625516	1.12	0.868643		
0.13	0.551717	0.33	0.6293	1.13	0.870762		
0.14	0.55567	0.34	0.633072	1.14	0.872857		
0.15	0.559618	0.35	0.636831	1.15	0.874928		
0.16	0.563559	0.36	0.640576	1.16	0.876976		
0.17	0.567495	0.37	0.644309	1.17	0.878999		
0.18	0.571424	**0.38**	**0.648027**	1.18	0.881		
0.19	0.575345	0.39	0.651732	1.19	0.882977		



$\mathrm{P}(Z \le z) = \Phi(z) = \text{shaded area}$

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

$$\mu = 29{,}576 \text{ ksi} \qquad \text{and} \qquad \sigma = 1{,}507 \text{ ksi}$$

3. $$\mathrm{P}(E \geq 29{,}000) = 1 - \mathrm{P}(E \leq 29{,}000) = 1 - \Phi\left[\frac{E-\mu}{\sigma}\right]$$
$$= 1 - \Phi\left[\frac{29{,}000 - 29{,}576}{1{,}507}\right]$$
$$= 1 - \Phi(-0.38) = 1 - [1 - \Phi(0.38)]$$
$$= 1 - [1 - 0.64803] = 0.64803$$

4. $$\Phi\left(\frac{E - 29{,}576}{1{,}507}\right) = 0.10 \quad \text{or} \quad \left(\frac{E - 29{,}576}{1{,}507}\right) = \Phi^{-1}(0.10) = -\Phi^{-1}(0.90) = -1.28$$
$$\therefore E = 29{,}576 - 1.28 \times 1507 = 27{,}647 \text{ ksi}$$

Common Continuous Probability Distributions

- Lognormal Distribution
  - Any random variable X is considered to have a lognormal distribution if $Y = \ln(X)$ has a normal probability distribution, where $\ln(x)$ is the natural logarithm to the base e.
  - In many engineering problems, a random variable cannot have negative values due to the physical aspects of the problem.
  - In this situation, modeling the variable as lognormal is more appropriate.

Common Continuous Probability Distributions

- Lognormal Distribution
  - The probability density function (PDF) for the lognormal distribution of a random variable X is given by

$$f_X(x) = \frac{1}{x\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu_Y}{\sigma}\right]^2} \quad \text{for } 0 < x < +\infty$$

It is common to use the notation $X \sim \mathrm{LN}\left(\mu_Y, \sigma_Y^2\right)$. The notation states that X is lognormally distributed with a parameters μ_Y and variance σ_Y^2.

Common Continuous Probability Distributions

- Lognormal Distribution

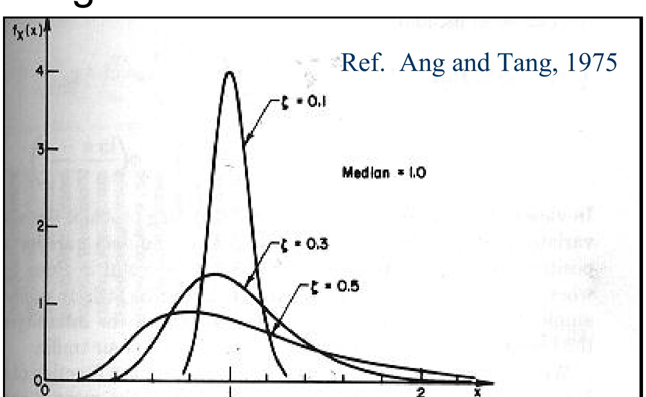


Figure 3.8 Log-normal density functions

Common Continuous Probability Distributions

- Properties of Lognormal Distribution
  1. The values of the random variable *X* are positive
  2. $f_X(x)$ is not symmetric density function about the mean value μ_X.
  3. The mean value μ_X and σ_X^2 are not equal to the parameters of the distribution μ_y and σ_Y^2 .
  4. They are related as shown in the next viewgraph.
  5. In many references, the notations λ_X and ζ_X are used in place of μ_Y and σ_Y^2 , respectively.

Common Continuous Probability Distributions

- Lognormal Distribution
  - Relationships between μ_X, μ_Y, σ_X^2 , and σ_Y^2

$$\sigma_Y^2 = \ln\left[1+\left(\frac{\sigma_X}{\mu_X}\right)^2\right] \quad \text{and} \quad \mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2$$

These two relations can be inverted as follows:

$$\mu_X = e^{\left(\mu_Y + \frac{1}{2}\sigma_Y^2\right)} \quad \text{and} \quad \sigma_X^2 = \mu_X^2\left(e^{\sigma_Y^2} - 1\right)$$

Note: for small *COV* or $\delta_X = \sigma_X / \mu_X < 0.3$, $\sigma_Y \approx \delta_X$

Common Continuous Probability Distributions

- Useful Properties of Lognormal Distribution

1. The multiplication of *n* lognormally distributed random variables $X_1, X_2, \ldots, X_n$ is a lognormal distribution with the following statistical characteristics:

$$W = X_1 X_2 X_3 \ldots X_n$$

The mean of W is

$$\mu_W = \mu_{Y_1} + \mu_{Y_2} + \mu_{Y_3} + \ldots + \mu_{Y_n}$$

Common Continuous Probability Distributions

The variance or second moment of W is

$$\sigma_W^2 = \sigma_{Y_1}^2 + \sigma_{Y_2}^2 + \sigma_{Y_3}^2 + \ldots + \sigma_{Y_n}^2$$

2. Central limit theorem: The multiplication of a number of individual random variables approaches a lognormal distribution as the number of the random variables approaches infinity. The result is valid regardless of the underlying distribution types of the random variables.

Common Continuous Probability Distributions

- Transformation to Standard Normal Distribution

$$Z = \frac{\ln X - \mu_Y}{\sigma_Y} \qquad P(X \le x) = \int_0^{+\infty} \frac{1}{x\sigma_Y\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)^2\right]} dx$$

Changing the variable,

$$P(X \le x) = \int_{-\infty}^{\frac{\ln x - \mu_Y}{\sigma_Y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)$$

It can be shown that

$$P(a \le X \le b) = F_X(b) - F_X(a) = \Phi\left(\frac{\ln b - \mu_Y}{\sigma_Y}\right) - \Phi\left(\frac{\ln a - \mu_Y}{\sigma_Y}\right)$$

Common Continuous Probability Distributions

- Example: Concrete Strength

A structural engineer of the previous example decided to use a lognormal distribution to model the strength of concrete. The mean and standard deviation are same as before, i.e., 3500 psi and 288.7 psi, respectively. What is the probability that the concrete strength is larger than 3600 psi?

$$\mu = 3500 \text{ psi} \qquad \text{and} \qquad \sigma = 288.7 \text{ psi}$$

Common Continuous Probability Distributions

- Example (cont'd): Concrete Strength

$$\sigma_Y^2 = \ln\left[1+\left(\frac{\sigma_X}{\mu_X}\right)^2\right] = \ln\left[1+\left(\frac{288.7}{3500}\right)^2\right] = 0.00678$$

$$\mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2 = \ln(3500) - \frac{1}{2}(0.00678) = 8.15713$$

The probability that the strength > 3600 psi :

$$P(X > 3600) = 1 - P(X \leq 3600) = 1 - \Phi\left[\frac{\ln x - \mu_Y}{\sigma_Y}\right]$$

$$= 1 - \Phi\left[\frac{\ln(3600) - 8.15713}{\sqrt{0.00678}}\right] = 1 - \Phi(0.3833)$$

$$= 0.3507$$

Common Continuous Probability Distributions

- Example (cont'd): Concrete Strength
  - The answer in this case is slightly different from the corresponding value (0.3645) of the previous example for the normal distribution case.
  - It should be noted that this positive property of the random variable of a lognormal distribution should not be used as the only basis for justifying its use.
  - Statistical bases for selecting probability distribution can be used as will be discussed later.

Common Continuous Probability Distributions

- Example: Modulus of Elasticity

The randomness in the modulus of elasticity (or Young's modulus) E can be described by a normal random variable. If the mean and standard deviation were estimated to be 29,567 ksi and 1,507 ksi, respectively,

1. What is the probability of E having a value between 28,000 ksi and 29,500 ksi?
2. The commonly used Young's modulus E for steel is 29,000 ksi. What is the probability of E being less than the design value, that is $E \lesssim$ 29,000 ksi?
3. What is the probability that E is at least 29,000 ksi?
4. What is the value of E corresponding to 10-percentile?

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

$$\mu_X = 29{,}576 \text{ ksi} \quad \text{and} \quad \sigma_X = 1{,}507 \text{ ksi}$$

$$\text{COV}(X) \text{ or } \delta_X = \frac{\sigma_X}{\mu_X} = \frac{1{,}507}{29{,}576} = 0.051 \leq 0.3$$

Therefore, $\sigma_Y \approx \delta_X = 0.051$

$$\mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2 = \ln(29{,}576) - \frac{(0.051)^2}{2} = 10.293$$

1. $$P(\leq 28{,}000 \leq E \leq 29{,}500) = \Phi\left(\frac{\ln(29{,}500) - 10.293}{0.051}\right) - \left(\frac{\ln(28{,}000) - 10.293}{0.051}\right)$$
$$= \Phi(-0.017) - \Phi(-1.04) = (1 - \Phi(0.017)) - (1 - \Phi(1.04))$$
$$= (1 - 0.50678) - (1 - 0.85083) = 0.34405$$

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

2. The probability of E being less than 29,000 ksi is

$$P(E \leq 29{,}000) = \Phi\left(\frac{\ln(29{,}000) - 10.293}{0.051}\right) = \Phi(-0.35) = 1 - \Phi(0.35)$$
$$= 1 - 0.63683 = 0.36317$$

3. The probability of E being at least 29,000 ksi is

$$P(E > 29{,}000) = 1 - P(E \leq 29{,}000)$$
$$= 1 - \Phi(-0.35) = 1 - (1 - \Phi(0.35))$$
$$= 1 - 1 + 0.63683 = 0.0.63683$$

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

4. For 10 - percentile, the E value will computed as follows:

$$\Phi\left(\frac{\ln(E) - 10.293}{0.051}\right) = 0.10$$

or

$$\left(\frac{\ln(E) - 10.293}{0.051}\right) = \Phi^{-1}(0.10) = -\Phi^{-1}(0.90) = -1.28$$

Thus,

$$\ln E = 10.293 - 1.28(0.051)$$

or

$$E = 27{,}659 \text{ ksi}$$

Common Continuous Probability Distributions

- Exponential Distribution
  – The importance of this distribution comes from its relationship to the Poisson distribution.
  – For a given Poisson process, the time T between the consecutive occurrence of events has an exponential distribution.
  – This distribution is commonly used to model earthquakes.

Common Continuous Probability Distributions

- Exponential Distribution
  - The probability density function (PDF) for the exponential distribution of a random variable T is given by

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

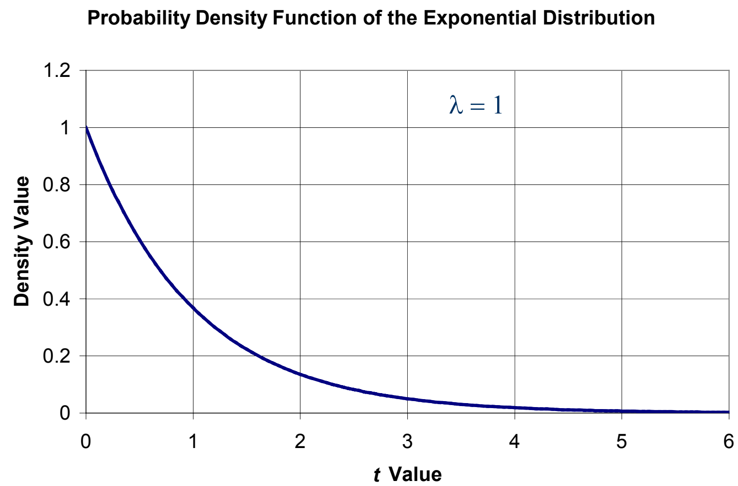
The cumulative distribution function is given by

$$F_T(t) = 1 - e^{-\lambda_t}$$

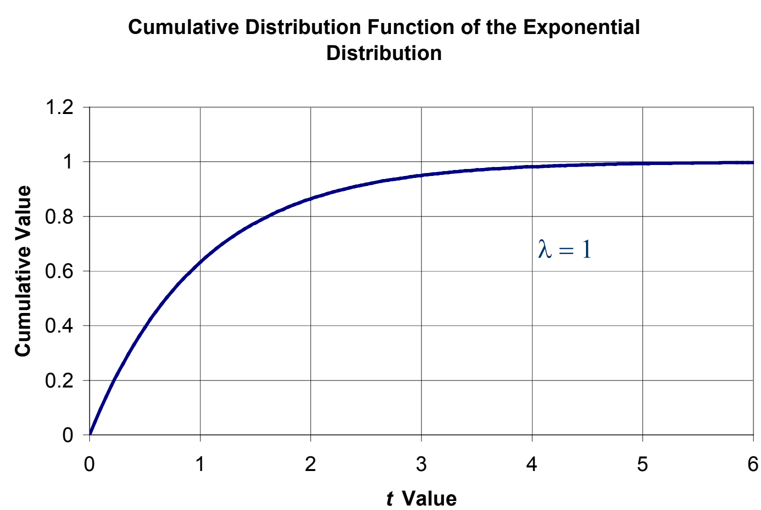
The mean value and the variance are given, respectively, by

$$\mu_T = \frac{1}{\lambda} \quad \text{and} \quad \sigma_T^2 = \frac{1}{\lambda^2}$$

Common Continuous Probability Distributions



Common Continuous Probability Distributions



Common Continuous Probability Distributions

- Exponential Distribution
  - Return Period

Based on the means of the exponential and Poisson distributions, the mean recurrence time (or return period) is defined as

$$\boxed{\text{Return Period} = \frac{1}{\lambda}}$$

Common Continuous Probability Distributions

- Example: Earthquake Occurrence

Historical records of earthquake in San Francisco, California, show that during the period 1836 – 1961, there were 16 earthquakes of intensity VI or more. What is the probability that an earthquake will occur within the next 2 years? What is the probability that no earthquake will occur in the next 10 years? What is the return period of an intensity VI earthquake?

Common Continuous Probability Distributions

- Example (cont'd): Earthquake Occurrence

$$\lambda = \frac{\text{Number of Earthquakes}}{\text{Number of Years}} = \frac{16}{1961-1816} = 0.128 \text{ per year}$$

The probability that an earthquake will occur within the next 2 years is

$$\mathrm{P}(T \leq 2) = 1 - e^{-\lambda t} = 1 - e^{-(0.128)(2)} = 0.226$$

The probability that no earthquake will occur in the next 10 years is

$$\mathrm{P}(T > 10) = 1 - F_T(10) = 1 - \left(1 - e^{-10\lambda}\right) = e^{-10(0.128)} = 0.278$$

The return period is given by

$$\text{return period} = E(T) = \frac{1}{\lambda} = \frac{1}{0.128} = 7.8 \text{ years}$$