CHAPTER

11b

Duxbury
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Learning

Making Hard Decision

Third Edition

MONTE CARLO SIMULATION

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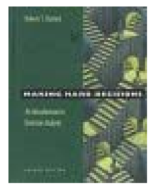

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ENCE 627 – Decision Analysis for Engineering

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Simulation

Methodology of Modeling Uncertainty

The Methodology of Modeling Uncertainty is described in five chapters that mainly concentrating on how to model uncertainty using probabilities and information as follows:

Detailed Steps

- Probability Basics: reviews fundamental probability concepts. — *Chapter 7*
- Subjective probability: translates beliefs & feelings about uncertainty in probability for use in decision modeling. — *Chapter 8*
- Theoretical Probability Models: helps with representing uncertainty in decision modeling — *Chapter 9*
- Using Data: uses historical data for developing probability distributions — *Chapter 10*
- **Monte Carlo Simulation: to give the decision-maker a fair idea about the probabilities associated with various outcomes.** — *Chapter 11*
- Value of Information: explores the value of information within the decision-analysis framework. — *Chapter 12*

Introduction to Simulation

- Simulation Techniques
  - Simulation is a process of replicating the real world based on a set of assumptions and conceived models of reality.
  - Simulation can be performed either:
    - Experimentally, or
    - Theoretically
  - In practice, theoretical simulation is performed (inexpensive).

Introduction to Simulation

- Simulation may be applied in engineering to predict or study the performance and response of a system.
- Simulation can be used to verify the accuracy of structural reliability methods with little background in probability and statistics.
- A simulation method can provide estimates for any problem, whereas analytical methods may not always converge in their iterations

Introduction to Simulation

- Need for Simulation and Data Analysis
  - Examples:
    - Transportation engineers frequently use traffic counts at intersection or accident data for various configurations of control signals in designing highways. However, if these data are insufficient or costly, they resort to *simulation*.
    - Environmental engineers collect data on water quality and analyze these data to decide upon the type of water treatment that is needed. Unfortunately, stream-flow records often do not include extreme floods that are important in evaluating flood risk. For this reason, they use *simulated data* to help making decisions.

Introduction to Simulation

- Example: Simulation by Coin Flipping
  - Water quality for a particular location on a river
    - Assumptions
      - About 50% of the time acceptable
      - 50% of the time unacceptable
      - Data was obtained for the last two weeks (14 days)
        - » AAUAUUAUUAAAUA
      - Damage to aquatic life occurs if the water quality is unacceptable for three or more consecutive days

A = acceptable
U = unacceptable

Introduction to Simulation

- Conclusion
  - According to the real data, the water quality was of acceptable quality 8/14 = 0.57 (57%) of the days and there were no instances of aquatic damage.

Should the engineer, therefore, believe that aquatic damage will not occur in the future? Of course not!

Introduction to Simulation

- Use of simulation to make a decision based on the probability of aquatic damage in the future
  - Flipping a coin 56 times produces the 8-week sequence:

    HHTHTHHHHHTHTHTTHTTTHTHHHTHT

    THTTTTHHTHHHTHTHTHTTHHHTTHTH

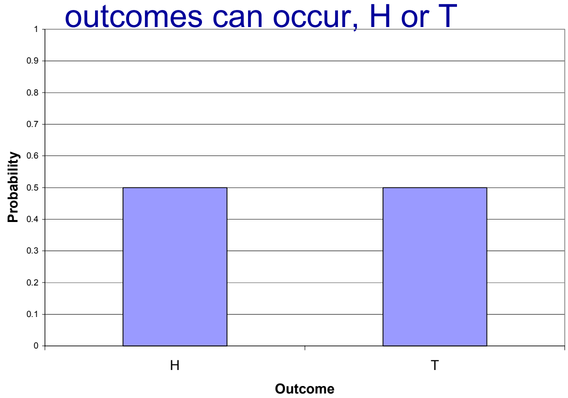
H = head (acceptable)
T = tail (unacceptable)


Introduction to Simulation

- Conclusions
  - If a tail is considered an acceptable water quality, then two occurrences of unacceptable quality happened in 8 weeks
  - This represents 4 weeks or 13 times a year

Generation of Random Numbers

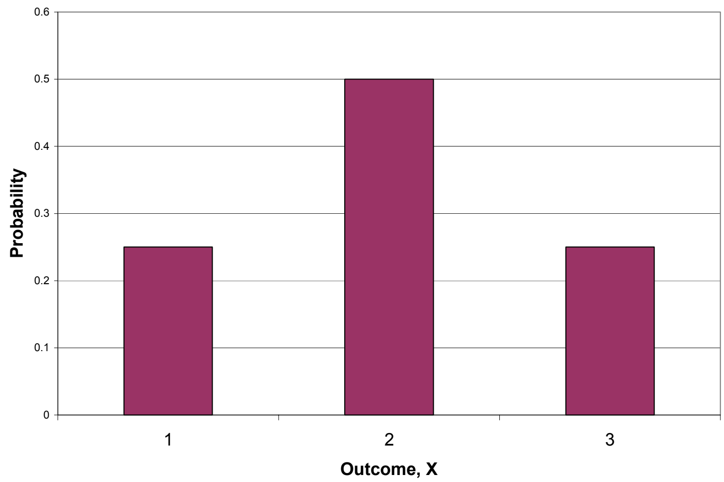
- Flip of a Coin
    - For one event, only one of the two possible outcomes can occur, H or T



Generation of Random Numbers

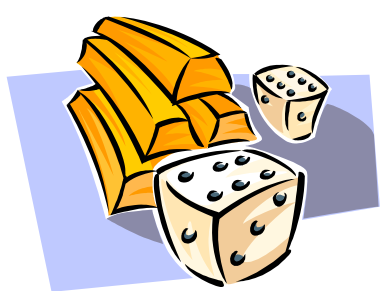
- Flipping of Two Coins
    - Outcome
        - (T,T), (H,T) or (T,H), and (H,H)
    - Let
        - (T,T) = 0
        - (H,T) or (T,H) = 1
        - (H,H) = 2
        - X = number of heads
    - Then, the probability $P(x)$ can be graphed as

Generation of Random Numbers



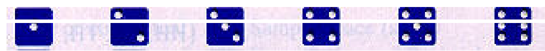
Generation of Random Numbers

- Rolling of Dice



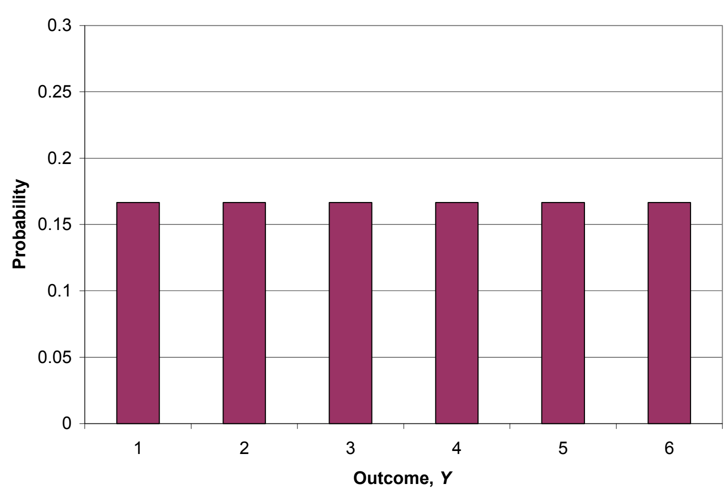
Generation of Random Numbers

- Rolling of a Single Die



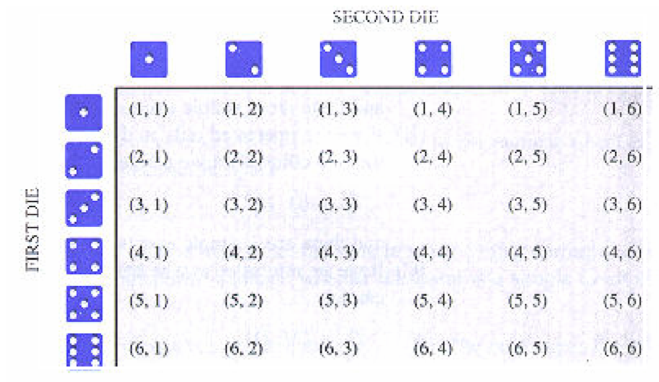
  – If the random events were generated with the roll of single die, one of the six outcomes is possible. If the die is fair, each outcome *y* is equally likely, and each would have a probability of 1/6.
  – Graphically, this can be presented as

Generation of Random Numbers



Generation of Random Numbers

- Rolling of Two Dice



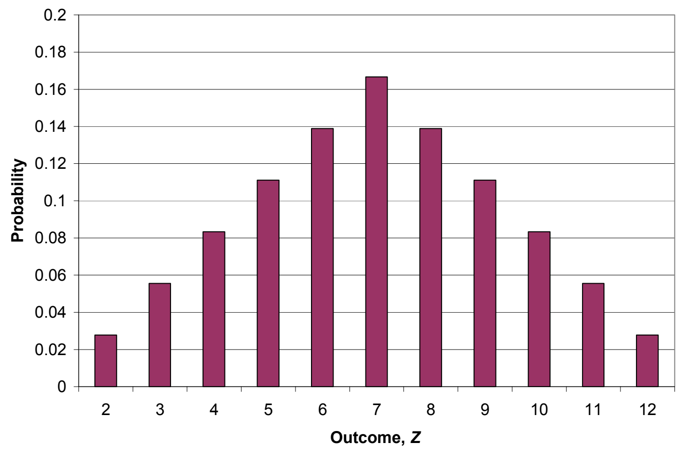
FIRST DIE \ SECOND DIE	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Generation of Random Numbers

- Rolling of Two Dice (cont'd)
  - If a pair of dice rolled simultaneously, the probability of the sum of dots from the two dice would appear graphically as shown in the following figure:

Generation of Random Numbers

Outcome	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36



Generation of Random Numbers

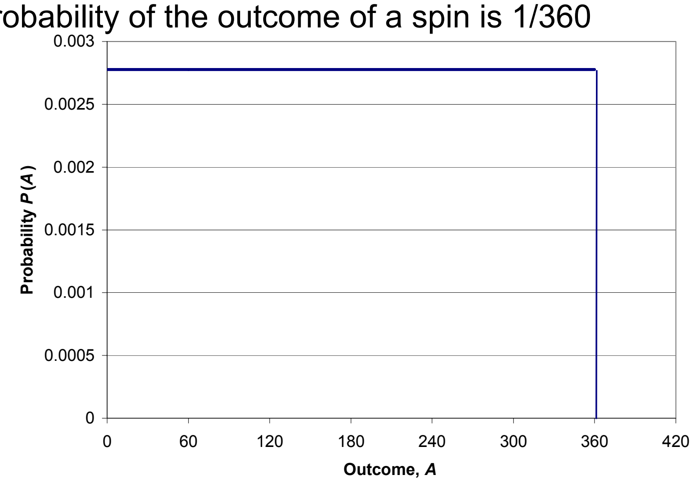
- Discrete versus Continuous Values
  - In each of the previous cases, only integers values were possible. For example, if a single die is rolled, a value of 4.6 is not possible.
  - Values from a flip of coin or a roll of a die are *discrete* (i.e., integers 1,2, …).
  - Examples: Engineering Cases:
    - Number of traffic fatalities
    - Number of floods per decade
    - Number of earthquakes above 6 per century

Generation of Random Numbers

- Discrete versus Continuous Values
  - Values on continuum could be generated with spinner (some board games) placed over a 3600 protractor.
  - Examples: Engineering Cases:
    - Stopping distance of as car
    - Magnitude of a flood
    - Compression strength of concrete
    - Weight of fertilizer used per acre

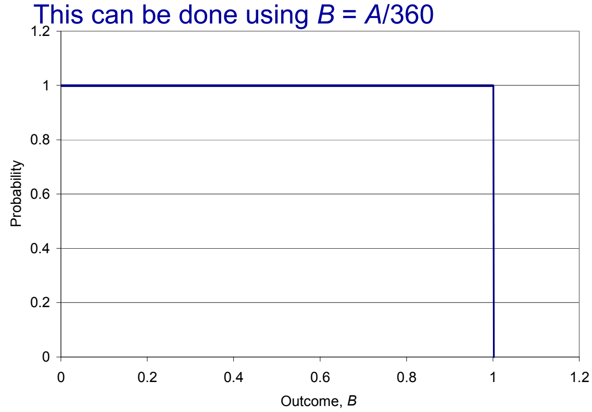
Generation of Random Numbers

Probability of the outcome of a spin is 1/360



Generation of Random Numbers

– Transformation of the angle A to a new variable B can be accomplished by requiring that B takes values from 0 to 1. This can be done using $B = A/360$



Computer Generation of Random Numbers

– Computer software packages are available

– The generated random numbers from these packages are called *pseudo random numbers*

– These numbers are generated from a well-defined and predictable process

Computer Generation of Random Numbers

- Midsquare Method
  - This method illustrates the problems associated with deterministic procedures
  - The general procedure is as follows:
    - Select at random a four-digit number (seed)
    - Square the number and write the square as an eight-digit number using preceding (lead) zeros if necessary
    - Use the four digits in the middle as the new random number.
    - Repeat steps 2 and 3 to generate as many numbers as necessary

Computer Generation of Random Numbers

- Example 1: Midsquare Method
  - Consider the seed number 2189. This value would produce the following:
  - 04791721
  - 62678889
  - 46076944
  - 00591361
  - 34963569
  - 92833225
  - 69422224

Computer Generation of Random Numbers

- Example 2: Midsquare Method
  – Consider the seed number 3500. This value would produce the following:
  – 12250000
  – 06250000
  – 06250000
  – 06250000
  – The above random-number sequence is not good for statistical purposes.
  – Other more reliable methods for generating random numbers are available.

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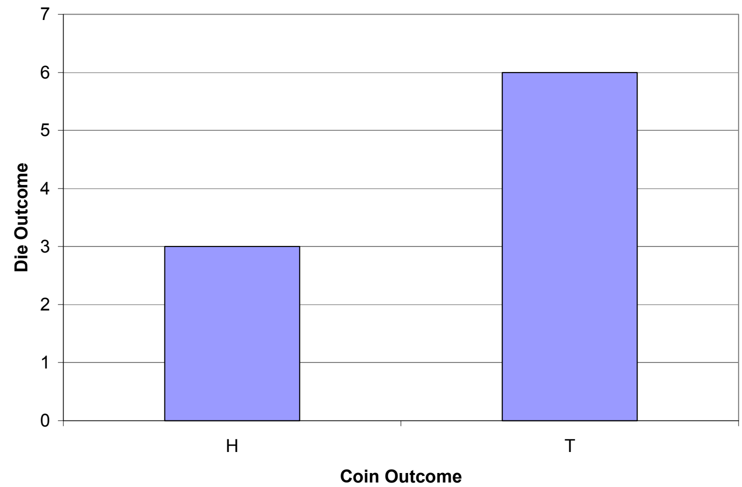
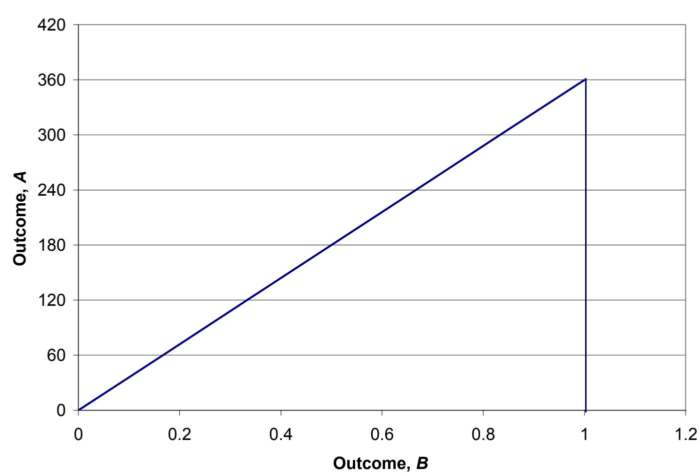
Transformation of Random Variables

- How can flips of a coin be generated with a die?
  – This can be accomplished by transforming the value of the die to the value of the coin.
  – The die is rolled and an occurrence of a 1, 2, or 3 would constitute a *head*, while 4, 5, or 6 would constitute a *tail.*
  – This can be presented graphically as

Transformation of Random Variables



Transformation of Random Variables



Simulation and Probability Distribution

- Need for Simulation
  - Simulation is used to verify the accuracy of structural reliability methods with little background in probability and statistics.
  - Measured data are often very limited, and making decision with small sample sizes increases the risk of incorrect decision.

Simulation and Probability Distribution

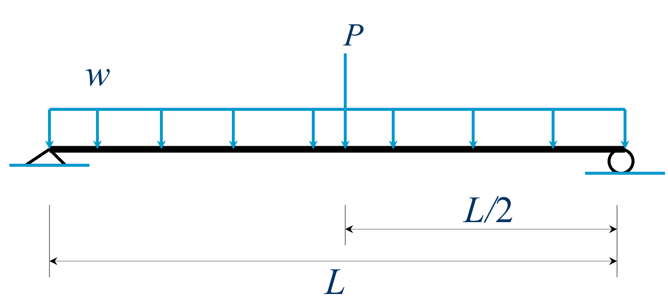
- Monte Carlo Simulation
  - Monte Carlo simulation has six essential elements:
    1. Defining the problem in terms of all the random variables,
    2. Quantifying the probabilistic characteristics of all the random variables (i.e., mean, *COV*, distribution type),
    3. Generating the values of these random variables

Simulation and Probability Distribution

- Monte Carlo Simulation (cont'd)
  4. Evaluating the problem deterministically for each set of all the random variables,
  5. Extracting probabilistic information from *n* observations.
  6. Determining the accuracy and efficiency of the simulation.

Simulation and Probability Distribution

- Formulation of the Problem
  – Consider a simply supported beam as shown



Simulation and Probability Distribution

- Assume both *w* and *P* are random variables.
- Thus, the design bending moment *M* at the midspan of the beam is also a random variable.
- The task now is to evaluate the probabilistic characteristics of the design bending moment using simulation.

Simulation and Probability Distribution

- If the span of the beam is 30 feet, the expression for the design moment can written as

$$M = \frac{wL^2}{8} + \frac{PL}{4}$$
$$= 112.5w + 7.5P$$

- *W* and *P* in this case are called basic random variables.

Simulation and Probability Distribution

- Generation of Random Variables
  – Computer software packages are available (e.g., Excel, Quattro Pro, etc.)
  – The generated random numbers from these packages are called *pseudo random numbers*
  – These numbers are generated from a well-defined and predictable process

Simulation and Probability Distribution

- Midsquare Method
  – This method illustrates the problems associated with deterministic procedures
  – The general procedure is as follows:
    1. Select at random a four-digit number (seed)
    2. Square the number and write the square as an eight-digit number using preceding (lead) zeros if necessary
    3. Use the four digits in the middle as the new random number.
    4. Repeat steps 2 and 3 to generate as many numbers as necessary

Simulation and Probability Distribution

- Example 1: Midsquare Method
  – Consider the seed number 2189. This value would produce the following:

04791721

62678889

46076944

00591361

34963569

92833225

69422224

Simulation and Probability Distribution

- Transformation of Uniform Random Numbers
  – The Uniform Distribution

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

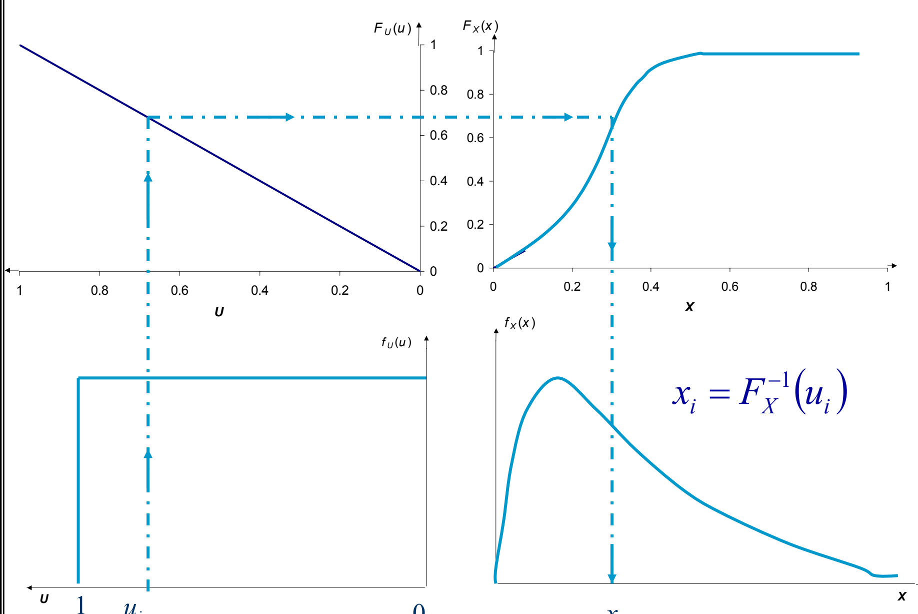
where $a < b$. The mean and variance are given by

$$\mu_X = \frac{a+b}{2} \quad \text{and} \quad \sigma_X^2 = \frac{(b-a)^2}{12}$$

Simulation and Probability Distribution

- Inverse Transformation Technique or Inverse CDF Method
  - In the in inverse transformation technique or inverse CDF method, the CDF of the random variable is equated to the generated random number *ui*, that is , *FX*(*xi*) = *ui*, and the equation can be solved for *xi* as follows:

$$\boxed{x_i = F_X^{-1}(u_i)}$$



Simulation and Probability Distribution

- Example: Normal Distribution
  - If X is normally distributed, that is $X \sim N(\mu,\sigma 2)$, then $Z = (X- \mu X)/\sigma X$ is a standard normal variate, that is, $Z \sim N(0,1)$. It can be shown that

$$u_i = F_X(x_i) = \Phi(z_i) = \Phi\left(\frac{x_i - \mu_X}{\sigma_X}\right)$$

or

$$z_i = \frac{x_i - \mu_X}{\sigma_X}$$

Thus,

$$x_i = \mu_X + \sigma_X z_i = \mu_X + \sigma_X \Phi^{-1}(u_i)$$

Simulation and Probability Distribution

- Example: Lognormal Distribution

If X is lognormally distributed, that is $X \sim LN(\mu,\sigma^2)$, then $Z = (\ln X- \mu_Y)/\sigma_Y$ is a standard normal variate, that is, $Z \sim N(0,1)$. It can be shown that

$$u_i = F_X(x_i) = \Phi(z_i) = \Phi\left(\frac{\ln x_i - \mu_Y}{\sigma_Y}\right)$$

or

$$\ln(x_i) = \mu_Y + \sigma_Y \Phi^{-1}(u_i)$$

Thus,

$$x_i = e^{\left[\mu_Y + \sigma_Y \Phi^{-1}(u_i)\right]}$$

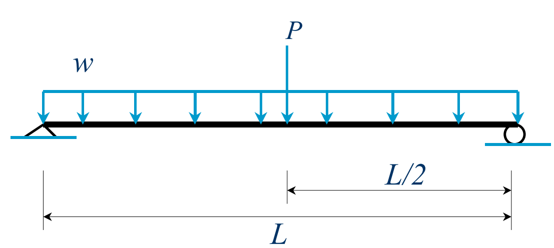
Simulation and Probability Distribution

- <u>Example:</u> Simply Supported Beam
  - The simply supported beam is subjected to the external loading *w* and *P* as shown in the figure. The probabilistic characteristics of the basic random variables are as follows:

Random Variable	Mean	COV	Standard Deviation	Distribution Type
L	30	-	-	Deterministic
w	2	0.10	0.2	Normal
P	20	0.15	3.0	Lognormal

Simulation and Probability Distribution

- <u>Example (cont'd):</u> Simply Supported Beam



$$M = \frac{wL^2}{8} + \frac{PL}{4} = 112.5w + 7.5P$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam
  1. Simulate the design moment M for 10 values.
  2. Also, Find the mean, variance, standard deviation, and coefficient of variation of M using the simulated sample values.

$$M = \frac{wL^2}{8} + \frac{PL}{4} = 112.5w + 7.5P$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam

Mean (w) =	2
Stdev (w) =	0.2

Mean (P) =	20
Stdev (P) =	3

u_1	u_2	w	P	M
0.388248947	0.874573	1.94322	23.47394542	394.6671
0.082540402	0.840615	1.72236	22.95014085	365.8919
0.891083258	0.540006	2.24646	20.07731226	403.3068
0.607604281	0.492971	2.05462	19.72681305	379.0954
0.682506093	0.103666	2.09494	16.38747866	358.5872
0.316169559	0.312569	1.90431	18.38852828	352.1491
0.949955696	0.726221	2.32889	21.63514737	424.2632
0.430819593	0.290549	1.96514	18.21599244	357.6985
0.697860999	0.904197	2.10365	24.0322073	416.9024
0.331143892	0.375488	1.91265	18.8642452	356.6548

Mean (M) =	380.9	kip-ft
Variance (M) =	727.5	kip-ft^2
Stdev (M) =	27.0	kip- ft
COV (M) =	0.071	

$$M = 112.5w + 7.5P$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam
  - Sample Calculations
    - Consider the second row in the table:
      - a) w: is normal

$$u_1 = F_W(w) = \Phi(z) = \Phi\left(\frac{w-\mu_W}{\sigma_W}\right)$$

or $$z = \frac{x-\mu_W}{\sigma_W}$$

Therefore,

$$w = \mu_W + \sigma_W z = \mu_W + \sigma_W \Phi^{-1}(u_1)$$
$$= 2 + 0.2\times\Phi^{-1}(0.08254) = 2 + 0.2\times -\Phi^{-1}(1-0.08254)$$
$$= 2 - 0.2\times\Phi^{-1}(0.91746) = 2 - 0.2(1.39) \approx 1.722$$

Simulation and Probability Distribution

- Sample Calculations

B) P: is lognormal $\sigma_Y = \sqrt{\ln\left[1+\left(\frac{\sigma_X}{\mu_X}\right)^2\right]} = \sqrt{\ln\left[1+\left(\frac{3}{20}\right)^2\right]} = 0.149$

$$\mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_X^2 = \ln(20) - \frac{1}{2}(0.149)^2 = 2.9846$$

$$u_2 = \Phi\left(\frac{\ln P - \mu_Y}{\sigma_Y}\right) \quad \text{or} \quad \ln P = \mu_Y + \sigma_Y \Phi^{-1}(u_2)$$

or

$$P = e^{\left[\mu_Y + \sigma_Y \Phi^{-1}(u_2)\right]} = e^{\left[2.9846 + 0.149\times\Phi^{-1}(0.84061)\right]}$$
$$= e^{\left[2.9846 + 0.149\times\Phi^{-1}(0.84061)\right]} = e^{\left[2.9846 + 0.149\times 1\right]} = 22.957$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam
  - Sample Calculations
    - Consider the the second row in the table
      - C) *M:*

$$M = 112.5w + 7.5P$$
$$= 112.5(1.722) + 7.5(22.957)$$
$$= 365.90 \text{ kip - ft}$$

Functions of Random Variables

- **Approximate Methods**
  - Taylor Series Expansion

A Taylor series is commonly used in engineering analysis to approximate functions that do not have closed form solution. The Taylor series is given by

$$f(x_0+h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + R_{n+1}$$

where

x_0 = base value or starting value

x = the point at which the value of the function is needed

$h = x - x_0$ = distance between x_0 and x (step size)

$n!$ = factorial of $n = n(n-1)(n-2)\ldots1$

$f^{(n)}$ = indicates the n^{th} derivative of the function $f(x)$

R_{n+1} = the remainder of Taylor series expansion

Functions of Random Variables

- Approximate Methods
  - Taylor Series Expansion
    - First-order approximation

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)$$

    - Second-order approximation

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2!}f^{(2)}(x_0)$$

    - Third-order approximation

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2!}f^{(2)}(x_0)+\frac{h^3}{3!}f^{(3)}(x_0)$$

Functions of Random Variables

- Approximate Methods
  - The Taylor series expansion can be used to approximate the mean and variance of a function of random variables $Y = g(\boldsymbol{X})$
  - Two cases to be considered:
    1. Single random variable X
    2. Multiple random variables, a random vector $\boldsymbol{X}$

Functions of Random Variables

- Approximate Methods
  - Single Random Variable *X*

    The Taylor series expansion of a function $Y = g(X)$ about the mean of X ($E(X)$) is given by

$$Y = g(X) = g(\mu_X + h) = g(\mu_X) + h\left.\frac{dg(X)}{dX}\right|_{\mu_X} + \frac{1}{2!}h^2\left.\frac{d^2g(X)}{dX^2}\right|_{\mu_X} + \ldots + \frac{1}{k!}h^k\left.\frac{d^kg(X)}{dX^k}\right|_{\mu_X}$$

or

$$Y = g(\mu_X) + [X - \mu_X]\left.\frac{dg(X)}{dX}\right|_{\mu_X} + \frac{1}{2}[X - \mu_X]^2\left.\frac{d^2g(X)}{dX^2}\right|_{\mu_X} + \ldots + \frac{1}{k!}[X - \mu_X]^k\left.\frac{d^kg(X)}{dX^k}\right|_{\mu_X}$$

Functions of Random Variables

- Approximate Methods
  - Single Random Variable *X*

$$g(X) = g[E(X)] + [X - E(X)]\left.\frac{dg(X)}{dX}\right|_{E(X)} + \frac{1}{2}[X - E(X)]^2\left.\frac{d^2g(X)}{dX^2}\right|_{E(X)}$$

$$+ \ldots + \frac{1}{k!}[X - E(X)]^k\left.\frac{d^kg(X)}{dX^k}\right|_{E(X)}$$

If the series is truncated at the second term, then

$$g(X) = g[E(X)] + [X - E(X)]\left.\frac{dg(X)}{dX}\right|_{E(X)}$$

Functions of Random Variables

- Approximate Methods
  - Single Random Variable *X*

$$g(X) = g[\mathrm{E}(X)] + [X - \mathrm{E}(X)]\left.\frac{dg(X)}{dX}\right|_{\mathrm{E}(X)}$$

Taking the expectation of both sides, and noting that

$$\mathrm{E}[X - \mathrm{E}(X)] = \mathrm{E}(X) - \mathrm{E}[\mathrm{E}(X)] = \mathrm{E}(X) - \mathrm{E}(X) = 0$$

Hence,

$$\boxed{\mathrm{E}(Y) = \mathrm{E}[g(X)] \approx g[\mathrm{E}(X)] \approx g(\mu_X)}$$

Functions of Random Variables

- Approximate Methods
  - Single Random Variable *X*

$$g(X) = g[\mathrm{E}(X)] + [X - \mathrm{E}(X)]\left.\frac{dg(X)}{dX}\right|_{\mathrm{E}(X)}$$

Taking the variances of both sides, and noting that

$$\mathrm{Var}[g[\mathrm{E}(X)]] = \mathrm{Var}[g(\mu_X)] = 0$$

Hence,

$$\boxed{\mathrm{E}(Y) = \mathrm{Var}\left[[X - \mathrm{E}(X)]\left.\frac{dg(X)}{dX}\right|_{\mathrm{E}(X)}\right] = \left[\left.\frac{dg(X)}{dX}\right|_{\mathrm{E}(X)}\right]^2 \mathrm{Var}(X)}$$

Functions of Random Variables

- Approximate Methods (single RV)
  - First-order (approximate) Mean

$$\mathrm{E}(Y) = \mu_Y = g[\mathrm{E}(X)]$$

  - First-order (approximate) Variance

$$\mathrm{Var}(Y) = \sigma_Y^2 = \left(\left. \frac{dg(X)}{dX} \right|_{\mathrm{E}(X)} \right)^2 \mathrm{Var}(X)$$

Functions of Random Variables

- Example: Pressure of Ocean Waves

The maximum impact pressure of ocean waves on coastal structures may be determined by

$$\rho_{\max} = 2.7 \frac{\rho K V^2}{D}$$

Where ρ = density of water, K = length of hypothetical piston, D = thickness of air cushion, V = horizontal velocity of advancing wave. Suppose that the mean crest velocity V is 4.5 ft/sec with *COV* of 0.2. ρ, K, and D are constants. If ρ = 1.96 slugs/cu ft, and the ratio K/D = 35, determine the mean and standard deviation of the peak impact pressure.

Functions of Random Variables

- Example (cont'd): Pressure of Ocean Waves

$$\mathrm{E}(\rho_{\max}) = \mu_{\rho_{\max}} \approx g[\mathrm{E}(V)] = 2.7(1.96)(35)(4.5)^2 = \underline{3750.7 \text{ psf}}$$

and

$$\left.\frac{d\rho_{\max}}{dV}\right|_{V=4.5} = \frac{d}{dV}\left(2.7\frac{\rho K V^2}{D}\right) = 2(2.7)\frac{\rho K V}{D} = 2(2.7)(1.96)(35)(4.5) = 1{,}666.98$$

$$\therefore \mathrm{Var}(\rho_{\max}) \approx \left(\left.\frac{d\rho_{\max}}{dV}\right|_{V=4.5}\right)^2 \mathrm{Var}(V) = (1{,}666.98)^2(0.2\times 4.5)^2 = 2{,}250{,}846.1 \text{ psf}^2$$

$$\therefore \sigma_{\rho_{\max}} = \sqrt{2{,}250{,}846.1} = \underline{1{,}500.3 \text{ psf}}$$

Functions of Random Variables

- Approximate Methods (Random Vector)
  - First-order (approximate) Mean

$$\mathrm{E}(Y) = \mu_Y = g[\mathrm{E}(X_1), \mathrm{E}(X_2), \ldots, \mathrm{E}(X_n)]$$

  - First-order (approximate) Variance

$$\mathrm{Var}(Y) = \sigma_Y^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial g(\boldsymbol{X})}{\partial X_i}\right|_{\mathrm{E}(X_i)} \left.\frac{\partial g(\boldsymbol{X})}{\partial X_j}\right|_{\mathrm{E}(X_i)} \mathrm{Cov}(X_i, X_j)$$

Functions of Random Variables

- Approximate Methods (Random Vector)
  – First-order (approximate) Variance

    If the X_i's are uncorrelated (statistically independent), then

$$\operatorname{Var}(Y) = \sigma_Y^2 \approx \sum_{i=1}^{n} \left(\left. \frac{\partial g(\boldsymbol{X})}{\partial X_i} \right|_{\mathrm{E}(X_i)} \right)^2 \operatorname{Var}(X_i)$$

Functions of Random Variables

- Example 1:

Assume that the random variable *Y* can be represented by the following relationship:

$$Y = X_1 X_2^2 X_3^{1/3}$$

where X_1, X_2, and X_3 are statistically independent random variables with mean values of 1.0, 1.5, and 0.8, respectively, and corresponding standard deviations of 0.1, 0.2, and 0.15, respectively. Find the first-order mean and standard deviation of *Y*.

Functions of Random Variables

- Example 1 (cont'd):

$$Y = X_1 X_2^2 X_3^{1/3}$$

$$\mathrm{E}(Y) = \mu_Y = g[\mathrm{E}(X_1), \mathrm{E}(X_2), \mathrm{E}(X_3)]$$
$$= (1.0)(1.5)^2(0.8)^{1/3} = 2.0887$$

$$\left.\frac{\partial Y}{\partial X_1}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(X_1 X_2^2 X_3^{1/3}\right)}{\partial X_1}\right|_{\mu_{X_i}} = \left.\left(X_2^2 X_3^{1/3}\right)\right|_{\mu_{X_i}} = \mu_{X_2}^2 \mu_{X_3}^{1/3}$$

$$\left.\frac{\partial Y}{\partial X_2}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(X_1 X_2^2 X_3^{1/3}\right)}{\partial X_2}\right|_{\mu_{X_i}} = \left.\left(2X_1 X_2 X_3^{1/3}\right)\right|_{\mu_{X_i}} = 2\mu_{X_1}\mu_{X_2}\mu_{X_3}^{1/3}$$

$$\left.\frac{\partial Y}{\partial X_3}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(X_1 X_2^2 X_3^{1/3}\right)}{\partial X_2}\right|_{\mu_{X_i}} = \left.\left(\frac{1}{3}\frac{X_1 X_2^2}{X_3^{2/3}}\right)\right|_{\mu_{X_i}} = \frac{\mu_{X_1}\mu_2^2}{3\mu_{X_3}^{2/3}}$$

Functions of Random Variables

- Example 1 (cont'd):

$$\left(\left.\frac{\partial Y}{\partial X_1}\right|_{\mu_{X_i}}\right)^2 = \left(\mu_{X_2}^2 \mu_{X_3}^{1/3}\right)^2 = \left[(1.5)^2(0.8)^{1/3}\right]^2 = 4.3627$$

$$\left(\left.\frac{\partial Y}{\partial X_2}\right|_{\mu_{X_i}}\right)^2 = \left(2\mu_{X_1}\mu_{X_2}\mu_{X_3}^{1/3}\right)^2 = \left[2(1)(1.5)(0.8)^{1/3}\right]^2 = 7.7560$$

$$\left(\left.\frac{\partial Y}{\partial X_3}\right|_{\mu_{X_i}}\right)^2 = \left(\frac{\mu_{X_1}\mu_2^2}{3\mu_{X_3}^{2/3}}\right)^2 = \left[\frac{(1)(1.5)^2}{3(0.8)^{2/3}}\right]^2 = 0.7574$$

Functions of Random Variables

- Example 1 (cont'd):

$$\mathrm{Var}(X_1)=\sigma_{X_1}^2=(0.1)^2=0.01$$

$$\mathrm{Var}(X_2)=\sigma_{X_2}^2=(0.2)^2=0.04$$

$$\mathrm{Var}(X_3)=\sigma_{X_3}^2=(0.15)^2=0.0225$$

$$\mathrm{Var}(Y)=\sigma_Y^2\approx\sum_{i=1}^{3}\left(\left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{\mathrm{E}(X_i)}\right)^2\mathrm{Var}(X_i)$$

$$\approx\left(\left.\frac{\partial g(Y)}{\partial X_1}\right|_{\mathrm{E}(X_i)}\right)^2\mathrm{Var}(X_1)+\left(\left.\frac{\partial g(Y)}{\partial X_2}\right|_{\mathrm{E}(X_i)}\right)^2\mathrm{Var}(X_2)+\left(\left.\frac{\partial g(Y)}{\partial X_3}\right|_{\mathrm{E}(X_i)}\right)^2\mathrm{Var}(X_3)$$

$$\approx(4.3627)(0.01)+7.7560(0.04)+0.7574(0.0225)=0.3709$$

$$\therefore\sigma_Y=\sqrt{0.3709}=\underline{0.609}$$

Functions of Random Variables

- Example 2:

The stress F in a beam subjected to an external bending moment M is

$$F=\frac{My}{I}$$

where y is the distance from the neutral axis of the cross section of the beam to the point where the stress is calculated, and I = is the centriodal moment of inertia of the cross section. Assume that M and I are random variables with means μ_M and μ_I, respectively, and variances σ_M and σ_I, respectively.

Functions of Random Variables

- Example 2(cont'd):

Determine the mean and variance of *F* based on first-order approximation.

$$\mu_F = \frac{\mu_M y}{\mu_I}$$

$$\left.\frac{\partial F}{\partial M}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(\frac{My}{I}\right)}{\partial M}\right|_{\mu_{X_i}} = \left.\left(\frac{y}{I}\right)\right|_{\mu_{X_i}} = \frac{y}{\mu_I}$$

$$\left.\frac{\partial F}{\partial I}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(\frac{My}{I}\right)}{\partial I}\right|_{\mu_{X_i}} = \left.\left(-\frac{My}{I^2}\right)\right|_{\mu_{X_i}} = -\frac{\mu_M y}{\mu_I^2}$$

Functions of Random Variables

- Example 2(cont'd):

$$\mathrm{Var}(F) = \sigma_F^2 \approx \sum_{i=1}^{2}\left(\left.\frac{\partial g(\boldsymbol{X})}{\partial X_i}\right|_{\mathrm{E}(X_i)}\right)^2 \mathrm{Var}(X_i)$$

$$\approx \left(\left.\frac{\partial g(F)}{\partial M}\right|_{\mathrm{E}(X_i)}\right)^2 \mathrm{Var}(M) + \left(\left.\frac{\partial g(F)}{\partial I}\right|_{\mathrm{E}(X_i)}\right)^2 \mathrm{Var}(I)$$

$$\approx \left(\frac{y}{\mu_I}\right)^2 \sigma_M^2 + \left(\frac{\mu_M y}{\mu_I^2}\right)^2 \sigma_I^2$$

Multivariable Simulation

- Simulation can be used to study the probabilistic characteristics of a function of random variables.
- It can provides information about the distributions of random variables that is beyond the ability of theory.
- Theoretical relationships are often based on restrictive assumptions, such as normal distributions, that may not be valid for a given problem.

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - The stress at the extreme fibers of steel beam is given by

$$\sigma = \frac{Mc}{I}$$

  - To estimate the mean and standard deviation of σ, we can use the first-order approximation as discussed previously, regardless of the distribution types of the basic random variables *c* and *I*.

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - Simulation can also be used to study the probabilistic characteristics of σ, such as the mean and standard deviation.
  - However, the distribution types of the basic random variables *c* and *I* are required.

Multivariable Simulation

- Stress at Extreme Fibers of a Beam

Random Variable	Mean	Standard Deviation	Distribution Type
c	10	0.5	Normal
M	3000	900	Lognormal
I	1000	80	Normal

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - First-order Approximate mean and standard deviation of *M*

$$\mu_\sigma = \frac{\overline{M}\overline{c}}{\overline{I}} = \frac{3000(10)}{1000} = 30$$

$$\left.\frac{\partial \sigma}{\partial M}\right|_{\overline{X}_i} = \left.\frac{c}{I}\right|_{\overline{X}_i} = \frac{10}{1000} = 0.01, \quad \left(\left.\frac{\partial \sigma}{\partial M}\right|_{\overline{X}_i}\right)^2 = 0.0001$$

$$\left.\frac{\partial \sigma}{\partial c}\right|_{\overline{X}_i} = \left.\frac{M}{I}\right|_{\overline{X}_i} = \frac{3000}{1000} = 3, \quad \left(\left.\frac{\partial \sigma}{\partial c}\right|_{\overline{X}_i}\right)^2 = 9$$

$$\left.\frac{\partial \sigma}{\partial I}\right|_{\overline{X}_i} = -\left.\frac{Mc}{I^2}\right|_{\overline{X}_i} = -\frac{3000(10)}{(1000)^2} = -0.03, \quad \left(\left.\frac{\partial \sigma}{\partial M}\right|_{\overline{X}_i}\right)^2 = 0.0009$$

$$\text{Var}(\sigma) = (0.0001)(900)^2 + 9(0.5)^2 + 0.0009(80)^2 = 89.01$$

$$\text{Standard Deviation } (\sigma) = \sqrt{89.01} = 9.43$$

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - Simulation result for the mean and standard deviation of *M*

For 1000 simulation cycles,

$$\text{Mean of } \sigma = 30.15$$

$$\text{Standard Deviation of } \sigma = 9.67$$

# Cycles	u1	u2	u3	*M*	*c*	*I*	σ
1	0.902062	0.735778	0.290168	4200.533	10.31519	955.7684	45.33453
2	0.94779	0.350819	0.607922	4628.371	9.808445	1021.913	44.42368
3	0.458328	0.615189	0.984024	2786.548	10.14643	1171.601	24.13239
4	0.450338	0.312857	0.78368	2770.104	9.756117	1062.775	25.42915
5	0.978812	0.734305	0.822165	5214.236	10.31294	1073.892	50.07405
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
1000	0.708552	0.253279	0.665872	3376.147	9.667897	1034.283	31.55831

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  – Comparison Between Approximate Method and Simulation

$\sigma = \frac{Mc}{I}$	Approximation	Simulation
Mean of σ	30.00	30.15
Standard Deviation of σ	9.43	9.67